



M. J. de Gucht sculp.

Johannes Elard
De Civitat: Cestriæ
Ætat: suæ 58 An. Dom: 1706

THE
Young Mathematician's Guide.
 Being a PLAIN and EASIE
INTRODUCTION
 TO THE
Mathematicks.
Margaret In Five PARTS. *Well*
 V I Z.

- I. **Arithmetick**, Vulgar and Decimal, in all its useful Rules; With a General Method of Extracting the Roots of all Single Powers.
- II. **Algebra**, or Arithmetick in Species; wherein the Method of Raising and Resolving Equations is rendred Easie; and Illustrated with Variety of Examples, and Numerical Questions. Also the whole Business of Interest and Annuities, &c. fully and plainly handled; with several New Improvements.
- III. The **Elements** of **Geometry**, Contracted, and Analytically Demonstrated: With a New and Easie Method of finding the Circles, Periphery and Area to any assign'd Exactness, by one Equation only: Also a New Way of making Sines and Tangents.
- IV. **Conick Sections**, wherein the Chief Properties, &c. of the Ellipsis, Parabola, and Hyperbola, are clearly Demonstrated.
- V. The **Arithmetick** of **Infinities** Explained, and rendred Easie; with its Application to Superficial and Solid Geometry.

With an
APPENDIX of **Practical Gauging.**

By **JOHN WARD**, Philomath.
Heretofore Chief Surveyor and Gauger-General in the Excise.

L O N D O N:
 Printed for *A. Bettefworth* at the *Red-Lyon* on *London-Bridge*, and are to be sold by *Ja. Woodward* in *St. Christopher's Church-Yard*, near the *Royal-Exchange*. M DCC IX.

INTRODUCTION

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INTRODUCTION

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General Method of Extracting the Roots of All Single Equations, Which are Reduced to an Equal Power.

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APPENDIX



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Printed for A. Easton, at the Red-Lion or London-Printers, and
are to be sold by J. W. Johnson in St. Christopher's Church-Yard,
near the Royal-Exchange, MDCCLXIX.

To the Honourable
S^r. JOHN WENTWORTH of
*North Elm's Hall in the West
Riding of Yorkshire, Baronet.*

S I R,

TIS now many Years, since I first
laid a Scheme of such a familiar
*Introduction to the Mathematicks, as would
Encourage the younger Students to proceed
with Ease and Chearfulness in that kind of
Learning. But the Design had been ren-
dered Abortive, had not your Approbation
(when you took the Trouble to peruse
the Mauuscript) given me Courage to pur-
sue it; so that whatsoever is found worth the
taking Notice of in this Treatise, is wholly
owing to you; And therefore, as you have so
far Influenced this Undertaking, I here beg
Leave it may be presented to the World un-
der your Protection.*

*Here, Sir I might expatiate in your Praises,
and say a great deal of my own Knowledge,
both as to your own Industry and Acquirments,*

The Dedication.

as also the *Encouragement* you give to others in the pursuit of all kinds of useful *Learning*: But I am very sensible how *Unpleasing* it would be to you, should I proceed in that *Strain*, (so usual in *Dedications*) and therefore will not presume to offend your *Ears* with mentioning *Particulars*: Nor need I do it, your *Generosity* and *Natural Inclinations* of doing Good to every one, being so well known to all that know you.

Only, Sir, permit me to assure you, that I am infinitely *Pleas'd* with the *Opportunity* of making this publick *Acknowledgment*, of the great *Obligations* which you have been so kind to confer upon,

S I R,

Your Honour's most Humble,

and most Obedient Servant,

J. WARD,

THE PREFACE.

WHoever Publishes any Book, seems oblig'd by Custom to write a Preface to it; wherein it's expected that he should shew the Usefulness of the Subject he Writes upon; And the Motives which induc'd him to write upon that Subject.

To the First, I presume it needless to say any thing; *viz.* of the Usefulness and Advantages that accrues to Mankind from *Mathematical Learning* in general; And especially from those two Excellent *Sciences Arithmetick*, and *Geometry*; there being no *Business, Employment, or Trade* whatsoever, that can be well manag'd without some Knowledge of the *First*; Nor any *Curious Mechanick Arts* either Invented, or Improved without the Assistance of the *Latter*. And therefore 'tis no Wonder, that in all *Ages* so many *Ingenious* and Learned Persons have employed part of their time in compiling large *Volumes* upon those Subjects. But many of those Authors seem to pre-suppose or take it for granted, That their Reader hath made some Progress in that sort of Learning, before he attempt to peruse their Works: And not only that, but very few have Treated upon more than one or two Sciences in any small *Volume*.

It was these Considerations that put me (*many Years ago*) upon composing the ensuing *Treatise*, purely for the Use and Benefit of such as are wholly Ignorant of the very first *Rudiments* (*Or have not the least Notion*) of *Mathematicks*: And therefore I've begun at the very Foundation or first Principles, *viz.* with an Unit in *Arithmetick*, and a Point in *Geometry*; And from those I proceed gradually, Leading, as it were, the young Learner, step by step, with all possible Plainness, till he arrives at such things as will (*with a little Application*) either Qualify him for Business; or fit and prepare

The P R E F A C E.

pare him for higher Speculations, without the Trouble and Charge of other Books.

The Title Page hath given a short Account of the several Parts Treated of, which I shall not Inlarge upon, but leave the *Book* to speak for it self; and if it be not able to give Satisfaction to the Reader, I am sure all I can say here in its behalf, will never Recommend it: But this I may (*without Vanity*) presume to say, that whoever Reads it over, will find more in it than the Title doth promise, or perhaps he expects. 'Tis true indeed, the Dress is but Plain and Homely, it being wholly intended to Instruct, and not to amuse or Puzzle the young Learner with hard Words; nor is it my Ambitious Desire of being thought more Learned or Knowing than really I am, by writing in such obscure Terms as a Late Author has done; However in this I shall always have the Satisfaction, That I've sincerely Aim'd at what is Useful, altho' in one of the meanest Ways; 'Tis Honour enough for me to be accounted as one of the under Labourers in Clearing the Ground a little, and Removing some of the Rubbish that lies in the Way to Knowledge. How well I have performed that, must be left to proper Judges.

To be brief; As I am not sensible of any Fundamental Error in this Treatise; so I will not pretend to say it is without Imperfections, (*Humanum est errare*) which I hope the Reader will excuse and pass over, with the like Candor and Good Will that it was composed for his Use; By his hearty well Wisher

J. Ward.

London
October 10th, 1706.
from my House near
the Kings-gate in
Red-Lyon Fields.

 Note, For want of Room here, those few Errors that escap'd unseen, are placed at the End of the Book.

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AN INTRODUCTION TO THE Mathematicks.

PART I.

Præcognita.

THE Business of Mathematicks in all its Parts both Theory and Practice, is only to search out and determine the true Quantity either of Matter, Space, or Motion, according as occasion requires.

By Quantity of Matter is here meant the Magnitude or bigness of any visible thing; whose Length, Breadth and Thickness, may either be measured or estimated.

By Quantity of Space is meant the distance of one thing from another.

And by Quantity of Motion is meant the swiftness of any thing moving from one place to another.

The consideration of these according as they may be proposed, are the Subjects of the Mathematicks, but chiefly that of Matter.

Now the consideration of Matter, with respect to its Quantity, Form and Position, which may either be Natural, Accidental, or Designed, will admit of infinite Varieties; But all the Varieties that are yet known, or indeed possible to be conceived, are wholly comprised under the due Consideration of these two, Magnitude and Number, which are the proper Subjects of Geometry, Arithmetick and Algebra. All other Parts of the Mathematicks being only the Branches of these three Sciences, or rather their Application to particular Cases.

Geometry is a Science by which we search out and come to know, either the whole Magnitude, or some part of any proposed Quantity: and is to be obtained by comparing it with another known Quantity of the same kind, which will always be one of these, viz. **A Line** (or Length only) **A Surface**, (that is Length and Breadth) or **A Solid** (which hath Length, Breadth and Depth, or Thickness) Nature admitting of no other Dimensions but these three.

Arithmetick is a Science by which we come to know what Number of Quantities there are (either real or imaginary) of any kind, contained in another Quantity of the same kind: Now this Consideration is very different from that of Geometry, which is only to find out true and proper Answers to all such Questions as demand, how Long, how Broad, how Big, &c. But when we are to consider either of more Quantities than one, or how often one Quantity is contained in another, then we have recourse to Arithmetick, which is to find out true and proper Answers to all such Questions as demand how many, what Number, or Multitude of Quantities there are. To be brief, the Subject of Geometry is that of Quantity, with respect to its Magnitude only; and the Subject of Arithmetick is Quantities with respect to their Number only.

Algebra is a Science by which the most abstruse or difficult Problems either in Arithmetick or Geometry are Resolved and Demonstrated, that is, it equally interferes with them both; and therefore its promiscuously named, being sometimes called Specious Arithmetick, as by Harriot, Vieta and Doctor Wallis, &c. And sometimes its called Modern Geometry, particularly the Ingenious Captain Halley, giving this following Instance of the Excellence of our Modern Algebra, Writes thus, 'The Excellence of the Modern Geometry (saith he) is in nothing more evident, than in those full and Adequate Solutions it gives to Problems: Representing all the possible Cases at one view, and in one General Theorem many times comprehending whole Sciences; which deduced at length into Propositions, and demonstrated after the manner of the Ancients, might well become the Subjects of large Treatises: For whatsoever Theorem solves the most complicated Problem of the kind, does with a due Reduction reach all the Subordinate Cases. Of which he gives a notable Instance in the Doctrine of Dioptricks for finding the Foci of Optick Glasses universally (vid. Philosophical Transactions. Numb. 205.)

Thus you have a short and general Account of the proper Subjects of those three Noble and Useful Sciences, Arithmetick,

Geo.

Geometry and Algebra. I shall now proceed to give a particular account of each, and first of Arithmetick, which is the Basis or Foundation of all Arts, both Mathematick and Mechanick; and therefore ought to be well understood before the rest are medled withal.

C H A P. I.

Concerning the several Parts of Arithmetick, with the Definition of such Characters as are used in this Treatise.

A Rithmetick, or the Art of Numbring, is fitly divided into three distinct Parts, two of which are properly called *Natural*, and the third *Artificial*.

The first being the most plain and easiest, is commonly called *Vulgar Arithmetick* in whole Numbers; because every Unit or Integer concerned in it, represents one whole Quantity of some Species or thing proposed.

The second is that which supposes an Unit (and consequently the Quantity or thing represented by that Unit) to be Broken or Divided into equal Parts (either even, or uneven) and considers of them either as pure Parts, viz. Each less than an Unit, or else of Parts and Integers intermixt. And is usually called the Doctrin of *Vulgar Fractions*.

The Third, or Artificial Part, is called *Decimal Arithmetick*; being an Artificial Invention of managing Fractions or Broken Numbers, by a much more commodious and easie way than that of *Vulgar Fractions*: For the several Operations performed in Decimals, differ but little from those in Whole Numbers; and therefore it is now become of general use, especially in Geometrical Computations.

Arithmetick (in all its Parts) is performed by the various ordering and disposing of Ten Arabick Characters or Numerical Figures (which by some are called Digits.)

viz. { One Two Three Four Five Six Seven Eight Nine Cypher:
1 2 3 4 5 6 7 8 9 0

The use of these Characters is said to be first introduced into England near six hundred Years ago, viz. about the Year 1130. vide Doctor Wallis's Algebra, Page 12.

The first of these *Characters* is called *Unity*, and represents one of any kind of *Species* or *Quantity*. As one *World*, one *Star*, one *Man*, &c.

Viz. *Unity* is that by which every thing that is, is called one (*Euclid* 7. Def. 1.) and is the beginning of all *Numbers*. That is to say, *Number* is a *Multitude* of *Units*. *Euclid* 7. Def. 2.

For, one mere one, makes two; and one more one more one makes three, &c. Which is the first and chief Postulate, or rather Axiom to *Arithmetick*.

Viz. That $1+1=2$. $1+1+1=3$. $1+1+1+1=4$. and so on.

Nine of these *Figures* were thus composed of *Units*, and differently form'd to represent so many *Units* put together into one *Sum*, as was intended each should denote. Nine being the greatest *Number* of *Units* that was then thought convenient to be expressed by one single *Character*; the last of the *Ten* is only a *Cypher*, or (as some phrase it) a nothing, because of it self it signifies nothing; for if never so many *Cyphers* be added to, or *Subtracted* from, any *Number*, they can neither increase nor diminish that *Number*; but yet as a *Cypher* (or *Cyphers*) may be placed, the other *Figures* will become of different *Values* from what they were before, as will appear further on.

For the more convenient ordering of the aforefaid *Numerical Figures*, according to the several Varieties that happen in *Computations*; I do advise the young Learner to acquaint himself with the Signification of the following *Algebraick Signs* or *Characters*, which he will find of excellent use, as being a much shorter, better and more significant way of denoting what is to be done (in most *Operations*) than can otherwise be expressed in words at length.

Significations.

Signs	Names.	The Sign of <i>Addition</i>; As $8+7$ is 8 more 7, and signifies that the <i>Numbers</i> 8 and 7 are to be added into one <i>Sum</i>. The like is to be understood when several <i>Numbers</i> are connected together with the Sign $+$. As $34+22+9+45$, &c. denotes these are all to be added into one <i>Sum</i>.
$+$	Plus or more.	

$-$ } } *Minus* } The Sign of *Subtraction*; As $9-6$, is 9 less
 } } *or less.* } 6, and signifies that 6, is to be taken from
 } } } 9, that so their difference may be found.

$\times$ } } *Into or* } The Sign of *Multiplication*; as 9×6 , is 9
 } } *with.* } into 6, and signifies that 9 is to be Multi-
 } } } plied into or with 6.

$\div$ } } *By.* } The Sign of *Division*; as $8 \div 2$, is 8 by 2,
 } } } and signifies that 8 is to be divided by 2, al-
 } } } so thus 2) 8 (4. or thus $\frac{8}{2}$ each signifying
 } } } the same thing, to wit, 8 divided by 2.

$=$ } } *Equal.* } The Sign of *Equality* or *Equation*, viz.
 } } } whenever the Sign $=$ is placed betwixt
 } } } *Numbers* (or *Quantities*) it denotes them
 } } } to be equal, as $9=9$, or $9+6=15$, or
 } } } $9-6=3$, &c. That is 9 is equal to 9, or 9
 } } } more 6, is equal to 15, and 9 less 6, is equal
 } } } to 3, &c.

$::$ } } *So is.* } The Sign of *Proportion*, or that common-
 } } } ly called the *Golden Rule*, or *Rule of Three*,
 } } } and $::$ is always placed betwixt the two
 } } } middle *Terms* or *Numbers* in proportion.
 } } } Thus $2 : 8 :: 6 : 24$. To be read thus ;
 } } } As 2, Is to 8, So is 6, To 24.

These *Signs* and their *Significations*, being perfectly learnt, will help to shorten the Work.

CHAP. II.

Concerning the Principal Rules in *Arithmetick*, and how they are performed in *Whole Numbers*.

THE Rules by which *Numerical Operations* are performed in all the Parts of *Arithmetick*, are many and various, several of them being form'd and rais'd as occasion requires, when applied to *Practice*, yet they are all comprehended within the due Consideration of these Six, viz.

One

The

Numeration (or Notation) Addition, Subtraction, Multiplication, Division and Evolution or Extraction of Roots.

Sect. I. Of Numeration, or Notation.

Numeration or Notation, Teacheth to Read or Express the true value of any *Number* when writ down; and consequently to write down any proposed *Number* according to its true value when it is named: And this consisteth of two *Parts*;

1. The due order of placing down *Figures*.
2. The true valuing of each *Figure* in its place.

Both which are plainly exhibited in the following *Table*.

<i>And is the first place in Counting.</i>									
Units	Tens	Hundreds	Thousands	Tens of Thousands	Hundreds of Thousands	Millions	Tens of Millions	Hundreds of Millions	Thousands of Millions
1	2	3	4	5	6	7	8	9	6 7 8
<i>Period of Units.</i>									
<i>Period of Thousands.</i>									
<i>Period of Millions.</i>									
<i>Period of Thousands of Millions.</i>									

By this **Numeration Table** it's apparent that the order of Places is reckoned from the right Hand towards the left; the first place of any *Number* being always that which is the outmost *Figure* to the right Hand; and whatever *Figure* stands in that place doth only signifie its own simple value, viz. So many *Units* as that *Figure* represents.

The second place is that of *Tens*, any *Figure* standing in that place signifies so many *Tens* as that *Figure* represents.

Units.

Units. The third place is *Hundreds*, the fourth place *Thousands*, &c. That is, each place towards the left Hand is *Ten* times the value of the next it towards the Right.

For instance, suppose 759 were proposed to be read or pronounced according to the value of each *Figure* as they now stand. The first *Figure* in this *Sum* is 9, because it stands in the place of *Units*, and therefore signifies but its own simple value, to wit 9 *Units* or *Nine*. The second *Figure* 5, stands in the place of *Tens*, and therefore signifies five *Tens* or *Fifty*. The *Figure* 7, stands in the third place, or place of *Hundreds*, and therefore it signifies *Seven Hundred*, and the whole *Sum* is to be read or pronounced thus, *Seven Hundred Fifty Nine*. Note, although the *Figure* 7, stands in the third place (according to the order of *Numbering*) yet when the whole *Sum* comes to be read it's first pronounced, the reading of *Numbers* being perform'd like that of Letters or Words, always beginning with the outmost *Figure* towards the left Hand, and so many *Figures* as are placed together without any Point, Comma, Line, or other Note of Distinction between them, are all but one *Sum*, and must be read as such.

For Example, 763596 is but one intire *Sum* or *Number*, notwithstanding it consists of six places of *Figures*, and is thus read; *Seven Hundred Sixty Three Thousand Five Hundred Ninety Six*.

The like is to be observed in reading or expressing the true value of any *Sum* or *Rank* of *Numbers* consisting of *Seven*, *Eight*, *Nine*, or more places of *Figures*, each *Figure* being to be valued according to its distance from the place of *Unity*: As in the foregoing Table.

Now such Values may as well arise by *Cyphers*, as by other *Figures*; for instance, 6, standing by it self, represents but six *Units*: But if a *Cypher* be annex to it thus, 60, then it becomes *Sixty*; for the *Cypher* possessing the place of *Units*, hath thereby removed the 6 into the place of *Tens*; and another *Cypher* more would make it 600, *Six Hundred*, &c.

Whence it may be noted, that altho' a *Cypher* of it self signifie nothing (as hath been said before) yet being placed on the right Hand of any *Figure*, it augments the value of that *Figure* by advancing it into a higher place than otherwise it would have been, had not the *Cypher* been there.

Take one Example more in *Numeration*, if you please, that in the Table, viz. 678987654321, which is, according as is there signified.

Six Hundred Seventy Eight Thousand Millions.
Nine Hundred Eighty Seven Millions.
Six Hundred Fifty Four Thousand.
Three Hundred Twenty One Units. Of any proposed
Species or Quantities whatsoever.

And here it may be observed, that every third *Figure* from the place of *Units*, bears the name of *Hundreds*; which shews that if any great *Sum* be parted, or rather distinguished into *Periods*, of three *Figures* in each *Period* (as in the foregoing *Table*) it will be of good use to help the young *Learner* in the easier valuing and expressing that *Sum*.

Sect. 2. Of Addition.

Postulate or Petition.

That any given *Number* may be increased or made more, by putting another *Number* to it.

Addition is that *Rule* by which several *Numbers* are collected and put together, that so their *Sum* or *Total* amount may be known.

In this *Rule* two things being carefully observed, the *Work* will be easily performed.

1. The first is the true placing of the *Numbers*, so as that each *Figure* may stand directly underneath those *Figures* of the same value, viz. Place *Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c.

Then underneath the lowest *Rank* (always) draw a *Line* to separate the given *Numbers* from their *Sum* when it's found.

Example. If these *Numbers* 54327, and 6251 were given to be added together, they must be placed,

Thus, $\begin{array}{r} 54327 \\ 6251 \\ \hline \end{array}$

2. The second thing to be observed is the due *Collecting* or *Adding* together each *Row* of *Figures* that stand over one another of the same value; and that is thus performed.

Rule. Always begin your *Addition* at the place of *Units*, and add together all the *Figures* that stand in that place, and if their *Sum* be under *Ten*, set it down below the *Line* underneath its own place; but if their *Sum* be more than *Ten*, you must set down only the overplus, or odd *Figure* above the *Ten* (or *Tens*) and so many *Tens* as the *Sum* of those *Units* amount

to, you must carry to the place of Tens; Adding them and all the Figures that stand in the place of Tens together, in the same manner as those of the Units were Added; then proceed in the same order to the place of Hundreds, and so on to each place until all is done.

The Sum arising from those Additions will be the Total Amount required.

Example 1.

Let it be required to find the Sum of the aforesaid Numbers:

viz. $\begin{array}{r} 54327 \\ 2651 \end{array}$

$\underline{56978}$ the Sum required.

Beginning at the place of Units, I say, 1 and 7 is 8, which being less than 10, I set it down (according to the Rule) underneath its own place of Units; and then proceed to the place of Tens, saying 5 and 2 is 7, which being less than 10, I set it down underneath its own place of Tens, and proceed to do the like at the place of Hundreds, and then at Thousands, setting each of their Sums underneath their own respective places: Lastly, because there is not any Figure in the lower Rank to be added to the Figure 5, which stands in the place of Ten Thous. in the upper Rank, I therefore bring down the said 5 to the rest, placing it underneath its own place, and then I find that $54327 + 2651 = 56978$, the true Sum required.

Example 2.

Suppose it were required to find the Sum of these Numbers, $3578 + 496 + 742 + 184 + 95$. These being placed, as before directed, will stand as in the Margin. Then beginning (as before) at the place of Units, say 8 and 4 is 12, and 6 is 18, and 2 is 20, and 5 is 25, set down the 5 Units underneath their own place of Units, and carry the 20, or two Tens to the place of Tens (at which place they are only 2) saying, 2 and 9 is 11, and 8 is 19, and 4 is 23, and 9 is 32, and 7 is 39, set down the 9 underneath its own place of Tens, and carry the 30, or three Tens (which indeed is 300) to the place of Hundreds, at which place they are but 3, saying, 3 I carry and 1 is 4, and 7 is 11, and 4 is 15, and 5 is 20, here because there is no Figure overplus (as before) I set down a Cypher underneath the place of Hundreds, and

carry

carry the 2 *Tens* (or rather the 2000) to the place of *Thousands*, saying (as before) 2 I carry and 3 is 5, which being the last, I set it down underneath its own place, and all is finished. And find the *Sum* or *Total* amount to be $5095 = 3578 + 496 + 742 + 184 + 95$.

If this Example be well considered, it will be sufficient to shew the usual Method of *Addition* in whole *Numbers*; but to make all plain and clear, I shall shew the young Learner the reason of carrying the *Tens* from one Degree or Row of *Figures*, to the next Superior Degree, which is done purely to save Trouble, and prevent the using of more *Figures* than are really necessary, as will appear by the following Method of adding together the same *Numbers* of the last Example.

Thus, Add together each single Row of *Figures* by it self; as if there were no more but that one Row, setting down the *Sum* underneath its own place.

3	5	7	8
4	9	6	
7	4	2	
1	8	4	
9	5		

The *Sum* of the Row of *Units*, is

2:5

The *Sum* of the Row of *Tens*, is

3:7:0

The *Sum* of the Row of *Hund.* is

1:7:0:0

The three *Thousands* brought down

3:0:0:0

} Add

The *Sum* or *Total* amount as before, is 5 0 9 5

From hence I presume it will be easie to conceive the true reason of carrying the foresaid *Tens*; and also that *Cyphers* do not augment or increase the *Sum* in *Addition*. (See Page 4.)

I might have here inserted a Lineal Demonstration of this Rule of *Addition*; but I thought it would rather puzzle than improve a young Learner, especially in this place; besides the reason of it is sufficiently evident from that Natural Truth of the whole being equal to all its parts taken together. Euclid 1. Axiom 19.

That is, the *Numbers* which are proposed to be added together are by that *Axiom* understood to be the several parts, and their *Sum* or *Total* amount found by *Addition*, is understood to be the whole.

And from thence is deduced the Method of proving the truth of any Operation in *Addition*, viz. By parting or separating the given *Numbers* into two Parcels (or more according to the largeness of it) and then Ad-

ding

ding up each Parcel by it self: For if those particular *Sums* so found, be *Added* into one *Sum*, and that *Sum* prove equal or the same with the *Total Sum* first found, then all is right; if not care must be taken to discover and correct the Error.

Example.

	5647	} The <i>Sum</i> of these Parts is, 12952
	3289	
	4016	
Add	2900	
	5007	} The <i>Sum</i> of these, is 9513
	1606	
The Tot. <i>Sum</i> of all these Parts	22465	} The <i>Sum</i> of each Parcel put together. 22465

Sect. 3. Of Subtraction.

Postulate or Petition.

That any *Number* may be diminished, or made less by taking another *Number* from it.

Subtraction is that *Rule* by which one *Number* is *Deducted* or taken out of another, that so the *Remainder*, *Difference*, or *Excess* may be known.

As 6 taken out of 9, there *Remains* 3. This 3 is also the *Difference* betwixt 6 and 9, or it is the *Excess* of 9 above 6.

Therefore the *Number* (or *Sum*) out of which *Subtraction* is required to be made, must be greater than (or at least equal to) the *Subtrahend* or *Number* to be *Subtracted*.

Note, This *Rule* is the *Converse* or direct contrary to *Addition*.

And here the same caution that was given in *Addition*, of placing *Figures* directly under those of the same value, viz. *Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c. Must be carefully observed, also underneath the lowest Rank there must be drawn a *Line* (as before in *Addition*) to separate the given *Numbers* from their *Difference* when it's found.

Then having placed the lesser *Number* under the greater, the *Operation* may be thus performed.

Rule. Begin at the Right Hand Figure or place of *Units* (as in *Addition*) and take or *Subtract* the lower Figure in that place from the Figure that stands over it, setting down the *Remainder* or *Difference* underneath its own place, if the

two Figures chance to be equal, set down a Cypher. But if the upper Figure be less than the lower Figure, then you must Add 10 to the upper Figure, or mentally call it 10 more than it is, and from that Sum Subtract the lower Figure, setting down the Remainder (as before directed.) Now because the 10 thus added, was suppos'd to be borrowed from the next Superior place (viz. of Tens) in the upper Figures, therefore you must either call the upper Figure in that place from whence the 10 was borrow'd, one less than really it is, or else (which is all one and most usual) you must call the lower Figure in that place one more than it really is, and then proceed to Subtraction in that place, as in the former; and so gradually on from one Row of Figures to another until all be done.

Example 1.

Let it be required to find the Difference between 6785 and 4572.

That is, let 4572 be Subtracted from 6785.

These Numbers being placed down, as before directed will stand.

$$\begin{array}{r} \text{Thus } \left\{ \begin{array}{l} 6785 \\ 4572 \\ \hline 2213 \end{array} \right. \end{array}$$

Beginning at the place of Units, take 2 from 5 and there will Remain 3, which must be set down underneath its own place, and then proceed to the place of Tens, taking 7 from 8, and there will Remain 1, to be set down underneath its own place; again at the place of Hundreds take 5 from 7, and there Remains 2, which set down, as before; lastly, take 4 from 6 and there will Remain 2, which being set down underneath its own place, the Work is finished, and the Difference so found will be $2213 = 6785 - 4572$ as was required.

Example 2.

The Difference between 5849 and 7496 is required.

Having placed the Numbers as in the Margin, begin at the place of Units (as before) and say 9 from 7496 6 cannot be, but 9 from 16 and there Remains 7, to 5849 1647 be set down under its own place, next proceed to the place of Tens, where you must now pay the 10 that was borrowed to make the 6, 16, by accounting the upper Figure 9 in that place one less than it is, saying, 4 from 8 and there Remains 4, or else (which is the most practised)

practised) say I borrowed and 4 is 5 from 9 and there *Remains* 4, to be set down under its own place (as before) again at the place of *Hundreds*, say 8 from 4 that cannot be, but 8 from 14 there will *Remain* 6 to be set down; and here I have borrowed 10 (as before) which must be paid in the same manner as the other 10 was, *viz.* either by calling the 7 in the upper Rank but 6, saying 5 from 6 there *Remains* 1, or else by saying I borrowed and 5 is 6 from 7 and there *Remains* 1, which being set down under its own place all is done, and the *Difference* required will be $1647 = 7496 - 5849$.

Example 3.

From 830476
Take 741068
Remains 89408

By this Example you may perceive that *Cyphers* in the *Subtrahend*, *viz.* in the *Numbers* to be *Subtracted*, do not diminish the *Number* from whence *Subtraction* is made. See Page 4.

These three Examples I presume may be sufficient to shew the young Learner the Method of *Subtracting* whole *Numbers*; as for the Reason thereof it's the same with that of *Addition*. Page 10. *viz.* of the whole being equal to all its parts taken together.

That is, in this Rule the *Number* from which *Subtraction* is required to be made, is understood to be the whole, and the *Subtrahend* or *Number* to be *Subtracted*, is suppos'd to be a part of that whole, consequently if that part be taken from the whole, the *Remainder* will be the other part.

From hence is deduced the common Method of proving *Subtraction*, by adding together the *Subtrahend* and the *Remainder*. For if the *Sum* of those two which are here called *Parts*, be equal to the *Number* from whence *Subtraction* was made (which is here called the whole) then the Work is right; if not, care must be taken to discover and correct the Error.

Example.

From 59435
Take 47608
Proof { 11827
 { 59435

Add

The Sum which is equal to the *Number* from whence *Subtraction* was made.

Or

Or from the abovesaid Reason, it will be easie to conceive how to prove the truth of *Subtraction* by *Subtraction*.

For, if from 59435 being here the whole,
there be taken 47608 as a part of that whole,
there will Remain 11827 the other part (as before)
And if from 59435 the whole, there be *Subtracted*.
the last part 11827
there will Remain 47608 the first part, or *Number* which
was required to be first *Subtracted*.

From	75643	From	7000000
Take	9000	Take	986432
	<u>66643</u>	Remains	<u>6013568</u>

Sect. 4. Of Multiplication.

Multiplication is a *Rule* by which any given *Number* may be speedily increased according to any proposed *Number* of *Times*.

That is, *One* *Number* is said to *Multiply* another, when the *Number* *Multiplied* is so often *Added* to it self, as there are *Units* in the *Number* *Multiplying*; and another *Number* is produced. (Euclid 7. Def. 15.)

To perform *Multiplication* there is required two given *Numbers*, called *Factors*.

The first is that *Number* which is to be *Multiplied*, and is generally put the greatest of the two *Numbers*, commonly called the *Multiplicand*.

The other is that *Number* by which the first is to be *Multiplied*, and is usually called the *Multiplicator* or *Multiplier*; and this denotes the *Number* of *Times* that the *Multiplicand* is required to be *Added* to it self. For so many *Units* as are contained in the *Multiplier*, so many times will the *Multiplicand* be really *Added* to it self. (as per Euclid above.) And from thence will arise a third *Number*, called the *Product*. But in Geometrical Operations its called the *Rectangle* or *Plain*.

For instance; Suppose it were required to increase 6 four times, That is, to *Multiply* 6, into or with 4, these two *Numbers* are to be set (or placed) down as in *Addition* or *Subtraction*.

Thus

Thus $\left\{ \begin{array}{l} 6 \\ 4 \end{array} \right.$ *Multiplicand, Multiplier,* } or *Factors.*

Product 24 viz. 4 times 6 is 24, as plainly appears by *Addition*, viz. By Setting down 6 four times, and then adding them together into one *Sum*, thus,

$$\begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{r} 6 \\ 6 \\ 6 \\ 6 \end{array} \left. \vphantom{\begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \end{array}} \right\} \text{Add}$$

From hence it is evident that Multiplication is only a Concise or Compendious way of Adding any given Number to it self so often as any Number of Times may be proposed.

Before any Operation can be readily Perform'd in Multiplication, the several *Products* of the single *Figures*, one into another must be perfectly learn'd by Heart, viz. That 2 times 2 is 4, that 3 times 3 is 9, that 3 times 6 is 18, &c. According as they are express'd in the following Table. Wherein I have omitted Multiplying with 2, it being so very easie that any one may do it.

Multiplication Table.

3×3=9	4×4=16	5×5=25	6×6=36	7×7=49	8×8=64
3×4=12	4×5=20	5×6=30	6×7=42	7×8=56	8×9=72
3×5=15	4×6=24	5×7=35	6×8=48	7×9=63	8×9=81
3×6=18	4×7=28	5×8=40	6×9=54		
3×7=21	4×8=32	5×9=45			
3×8=24	4×9=36				
3×9=27					

I think it needless to give any *Explanation* of this Table; for if the *Signs* and their *Significations* be well understood, (vide Page 5.) it must needs be easie. Only this may be noted, that $4 \times 3 = 3 \times 4$. or $7 \times 5 = 5 \times 7$, &c.

That is, 3 times 4, is the same with 4 times 3, or 5 times 7, is the same with 7 times 5, &c. The like must be understood of all the rest in the Table.

And when all these single *Products* are so perfectly Learn'd by Heart, as to be said without pausing; you may then proceed (but not till then) to the Business of *Multiplication*; which will be found very easie, if the following *Rule* (and *Examples*) be carefully observed.

Rule. Always begin with that Figure which stands in the Units place of the Multiplier; and with it Multiply the Figure which stands in the Units place of the Multiplicand; if

their

their Product be less than Ten, set it down underneath its own place of Units; and proceed to the next Figure of the Multiplicand. But if their Product be above Ten (or Tens) then set down the Overplus only (or odd Figure as in Addition) and bear (or carry) the said Ten or Tens in mind until you have Multiplied the next Figure of the Multiplicand, with the same Figure of the Multiplier; then to their Product Add the Ten or Tens beared in mind, setting down the overplus of their Sum above the Tens, as before; and so proceed on in the very same manner, until all the Figures of the Multiplicand are Multiplied with that Figure of the Multiplier.

Example 1.

Suppose it were required to Multiply 3213 into or with 3.

3213 Multiplicand, }
3 Multiplier, } or Factors.

Product 9639

Beginning at the Units place, say, 3 times 3 is 9, which because it is less than Ten, set it down underneath its own place, and proceed to the next place of Tens; saying 3 times 1 is 3, which set down underneath its own place, then to the next place, viz. of Hundreds, saying 3 times 2 is 6, which set down, as before, lastly, at the place of Thousands, say 3 times 3 is 9, which being set down underneath its own place, the Operation is finished; and the true Product is $9639 = 3213 \times 3$, as was required.

Example 2.

Let it be required to Multiply 8569 into 8. Set down these Numbers as before,

Thus $\begin{array}{r} 8569 \\ 8 \\ \hline 68552 \end{array}$

Beginning at the Units place say, 8 times 9 is 72, set down the 2 underneath its own place of Units, and bear the 70, or 7 Tens in mind, and proceed to the next Figure of the Multiplicand (at which place the 7 Tens are only 7) saying 8 times 6, is 48, and the 7 carried in mind is 55, set down the odd 5 underneath its own place of Tens, and carry the 50 (which is really 500) to the next place (viz. of Hundreds) at which place it's only 5, where say, 8 times 5, is 40; and the 5 carried in mind is 45, set down the 5 underneath its own place; and carry the 40 or 4 Tens (which

(which is really 4000) to the next place, viz. of *Thousands*, saying 8 times 8 is 64, and 4 carried in mind is 68. (Now this being the last place or *Figure* to be *Multiplied*) set down the whole *Product* 68, and the work is done.

So that, $8569 \times 8 = 68552$, the *Product* required.

Now the Reason of this and all other the like Operations, may be easily conceived from this which follows,

8569 } The same *Factors* as before.

72 } Here 8 times 9, is but 72, as before, because the 9 stands in the *Units* place.

480 } Now here it is not really 8 times 6 = 48, but it is 8 times 60 = 480, because the 6 stands in the place of *Tens*.

4000 } And here it is not 8 times 5 = 40, but it's really 8 times 500 = 4000, because the 5 stands in the place of *Hundreds*.

64000 } Lastly, because the 8 in the *Multiplicand* stands in the place of *Thousands*, it's therefore, 8 times 8000 = 64000, and not 8 times 8 = 64.

68552 } The *Sum* of the particular *Products*, which gives the true *Product*, as before.

By what hath been already said, with a little Consideration had to the *Examples*: I presume the Learner may easily understand how to *Multiply* whole *Numbers* with any single *Figure*. And when it is requir'd to *Multiply* with more than one; Then so many *Figures* as there are in the *Multiplier*, so many particular *Products* there must be.

That is, all the *Figures* of the *Multiplicand*, must be *Multiplied* with every single *Figure* of the *Multiplier* as if there were but one single *Figure*; and the *Sum* of all those particular *Products*, will be the true *Product* required. but in those Operations great care must be taken in setting down the particular *Products*, (which arise by each *Multiplying Figure*) in their proper places. And that will be easily done if the following *Directions* be carefully observ'd.

Always place the first *Figure* (or *Cypher*) of every particular *Product*, directly underneath the *Multiplying Figure*. Or thus:

The first *Figure* (or *Cypher*) of the second particular *Product* must stand directly under the second *Figure* (or place) of the first *Product*; and the first *Figure* (or *Cypher*) of the third particular *Product*, must stand directly underneath

the third Figure of the first Product: And so on until all is done.

Now the reason of placing the first Figure of every particular Product in this order, will be very obvious to any one that considers the last Example; wherein the Cyphers are only set down to shew the true distance of the first Figure in each particular Product from the Units place. And altho' it is not usual to set down Cyphers in this manner; yet they are always suppos'd to be there: That is, their places are always left void, as in the two following Examples; wherein I have placed Points instead of Cyphers.

Example 3.

Let it be required to Multiply 78094, into or with 7563.

78094 } Factors.
7563 }

234282	The first particular Product with	3
468564.	The second particular Product with	60
390470..	The third particular Product with	500
546658...	The fourth particular Product with	7000
590624922	The Total or true Product required.	

Example 4.

Suppose it required to Multiply 57498 into 60008

57498
60008

459984 The Product with 8
344988.... The Product with 60000

3450339984 = 57498 × 60008 as was required.

Here you may observe, that I pass over the Cyphers, and only take care of placing the first Product of the last Figure, viz. of 60000 according to the foregoing Directions.

When there is a Cypher, or Cyphers, to the Right-hand either of the Multiplicand, or Multiplier, or to both; in that case Multiply the Figures as before; neglecting the Cyphers until the particular Products are added together; Then to their Sum annex so many Cyphers as are in either or both the Factors. As in these,

Example 5.

Example 5.

$$\begin{array}{r}
 9538 \\
 4600 \\
 \hline
 57228 \\
 38152 \\
 \hline
 43874800.
 \end{array}$$

Example 6.

$$\begin{array}{r}
 87600 \\
 79 \\
 \hline
 7884 \\
 6132 \\
 \hline
 6920400.
 \end{array}$$

Example 7.

$$\begin{array}{r}
 785000 \\
 56900 \\
 \hline
 7065 \\
 4710 \\
 \hline
 3925 \\
 \hline
 44666500000.
 \end{array}$$

Take a few Examples without their work at large.

$$75649 \times 579 = 43800771$$

$$687000 \times 356 = 244572000$$

$$530674 \times 45007 = 23884044718$$

$$7901375 \times 30000 = 237041250000$$

$$537084000 \times 590700 = 317255518800000$$

$$102030405 \times 504030201 = 51426405540261405$$

$$987654321 \times 123456789 = 121932631112635269$$

Note, If it be required to Multiply any Number with 10, 100, 1000, 10000, &c. it's only Annexing the Cyphers of the Multiplier to the Figures of the Multiplicand, and the work is done.

$$\text{Thus } \begin{cases} 578 \times 10 = 5780 & 578 \times 1000 = 578000 \\ 578 \times 100 = 57800 & 578 \times 10000 = 5780000 \text{ \&c.} \end{cases}$$

These Examples (being well understood) are sufficient to instruct the Learner, in all the varieties that can happen in Multiplying of whole Numbers, according to the Method generally practised: However it may not be amiss to shew here how Multiplication may be performed (with many Figures) by Addition only.

Example.

Let it be required to Multiply 879654. into 79863.

In order to perform this (or any other Operation of this kind) by Addition only; you must make a Tariffa or small Table of the given Multiplicand, in this manner:

First, Make a small Column, and in it place gradually downward the nine single Figures; viz. 1. 2. 3. 4. 5. &c.

D 2

Then

Then against the *Figure 1*, set down the *Multiplicand* (which in this *Example* is 879654) and against the *Figure 2*, set down the double of the *Multiplicand*, found by *Adding* it to its self; To this double *Add* the *Multiplicand*, setting down their *Sum* against the *Figure 3*. And so proceed on by a continued *Addition* until there be Ten times the *Multiplicand* in the *Table*, which if the work is true will be the *Multiplicand* it self with a *Cypher* to the Right-hand of it (as in the Annexed Table) this being done it will be easie to conceive, that the *Figures* in the small Column of the *Table*, do respectively represent those of the *Multiplier*; And that the *Numbers* against any of those *Figures* in the small Column, will be the true *Product* of the *Multiplicand* agreeing to any *Figure* of the *Multiplier*; as plainly appears by the work of this *Example*.

1	879654
2	1759308
3	2638962
4	3518616
5	4398270
6	5277924
7	6157578
8	7037232
9	7916886
10	8796540

Then

879654 } the *Factors* as before.
 79863

Against 3, in the *Table* is 2638962 = 879654 × 3
 Against 6, is 5277924 = 879654 × 60
 Against 8, is 7037232 = 879654 × 800
 Against 9, is 7916886 = 879654 × 9000
 Against 7, is 6157578 = 879654 × 70000
 The *Product* required is 70251807402 = 879654 × 79863

Note, This Method of Tabulating the *Multiplicand*, is both easie and certain; being neither subject to Errors, nor burthensom to the Memory, and therefore in large Calculations it may be found very useful. But for common practice the useful Method (as in Page 18. &c.) is best, and to be prefer'd before this.

Most *Masters* that teach (and several *Authors* that write of) *Arithmetick* do teach to prove the truth of *Multiplication*, by casting away all the *Nines* that are contain'd in both the *Factors*, and their *Product*; but because that Method is very erroneious, as might be easily shew'd; I shall therefore omit inserting it, and leave the proof of *Multiplication* to the next *Section*, wherein (I presume) the Reason and proof, both of it, and *Division* will plainly appear.

Sect. 5. Of Division.

Division is a *Rule* by which one *Number* may be speedily *Subtracted* from another, so many times as it is contained therein.

That is, It speedily discovers how often one *Number* is contained (or may be found) in another: And to perform this there is required two *Numbers* to be given.

1. The one of them is that *Number* which is proposed to be *Divided*, and is called the *Dividend*.

2. The other is that *Number* by which the said *Dividend* is to be *Divided*, and is call'd the *Divisor*.

And by comparing these two, viz. the *Dividend* and *Divisor* together, there will arise a third *Number*, call'd the *Quotient*; which shews how often the *Divisor* is contained in the *Dividend*, or into what *Number* of Equal Parts the *Dividend* is then *Divided*. Therefore,

Division is by Euclid fitly term'd the measuring of one *Number* by another, viz. one *Number* is said to Measure another by that *Number*, which when it *Multiplies*, or is *Multiplied* by it, it produceth. Euclid 7 Defi. 23.

And, if a *Number* measuring another, Multiply that *Number* by which it Measureth, or be Multiplied by it, it produceth the *Number* which it Measureth. Euclid. 7. Ixiom. 9.

That is to say, If that *Number* which *Divides* another, (called the *Divisor*) be *Multiplied* with the *Number* which is produced by *Division* (called the *Quotient*) their *Product* will be the *Number* *Divided* or *Dividend*. Whence it follows that *Division* and *Multiplication* are the *Converse* or direct *Contrary* one to another (as *Subtraction* is to *Addition*) and do mutually prove the truth of each others *Operations*.

I shall therefore make choice of the foregoing *Examples* in *Multiplication*, in order (as I presume) to render the *Business* of *Division* more plain and easie.

First, Let it be required to find how often 6 is contained in 24. That is, to *Divide* 24 by 6.

N. B. Always place down the given *Numbers* in this order, first set down the *Divisor*, and to the right hand of it draw a crooked Line; then set down the *Dividend*, and to the right of it draw another crooked Line, in which must be placed the *Quotient Figure* or *Figures* as they become found. Thus,

Dividend.

Dividend

Divisor 6) 24 (4 the *Quotient*.

Here I consider how many times 6 there is in 24, and find it 4, viz. 4 times 6 is 24, therefore 4 is the true *Quotient* or *Answerer* required.

This is apparent by *Subtraction*, as in the Margin, where 24 the *Dividend* being set down, and from it 6, the *Divisor* is continually *Subtracted* so often as it can be, which is just 4 times. Therefore 4 is the true *Quotient* or *Answerer* required.

Compare this with the
Example, Page 15.

	24
1	6
	18
2	6
	12
3	6
	6
4	6
	0

Corollary.

From hence it is evident; that *Division* is but a *Concise* or *Compendious Method* of *Subtracting* one *Number* from another, so often as it can be found therein; for if the *Divisor* be continually *Subtracted* from the *Dividend*, accounting an *Unit* (or 1) for each time it is *Subtracted* (as above) the *Sum* of those *Units* will be the *Quotient*.

All Operations in *Division* do begin contrary to those of *Multiplication*, viz. At the first *Figure* to the left hand, or that of the highest value, and Decrease the *Dividend* by a repeated *Subtraction* of each *Product* arising from the *Divisor* when *Multiplied* into the *Quotient Figure*. And the only difficulty in *Division* of whole *Numbers* (or indeed of any *Numbers*) lies in making choice of such a *Quotient Figure*, as is neither too big nor too little; the which may be easily obtained by observing the following *Rule*, which hath two *Cases*.

Rule.

Case 1. So often as the first *Figure* of the *Divisor* is taken from the first *Figure* of the *Dividend*: So often must the second *Figure* of the *Divisor* be taken from the second *Figure* of the *Dividend*, when its joyn'd with what *Remains* of the first. And so often must the third *Figure* of the *Divisor* be taken from the third *Figure* of the *Dividend*, &c.

But if the first *Figure* of the *Divisor* cannot be taken from the first *Figure* of the *Dividend*. Then,

Case

Case 2. So often as the first Figure of the Divisor, is taken from the two first Figures of the Dividend, so often must the second Figure of the Divisor be taken from the third Figure of the Dividend, when it's joyn'd with what Remain'd of the second: And so often must the third Figure of the Divisor be taken from the fourth Figure of the Dividend, &c.

That is, the Quotient Figure must be such as being Multiplied into the Divisor, will Produce a Product equal to such a part of the Dividend as is then taken for that Operation: But if such a Product cannot be exactly found, then the next less must be taken, and ordered as in the following Examples; of which let that in Page 16 be the first, wherein there was given 8569 the Multiplicand, and 8 the Multiplier. To find the Product 68552. Let us here suppose the said Product 68552, and 8 the Multiplier, both given; thence to find the Multiplicand. That is, Let it be required to Divide 68552 by 8.

Dividend
Divisor 8) 68552 (Quotient when found.

According to the Rule, Case 1. I compare 8 the Divisor with 6 the first Figure of the Dividend, and finding I cannot take it from that, I then consider (by Case 2.) how often 8 can be taken from 68, the two first Figures of the Dividend, and find it may be taken 8 times; for 8 times 8 is 64, being the greatest Product of 8 (into any Figure) that can be taken from 68. I therefore place 8 in the Quotient, and with it Multiply 8 the Divisor, setting down their Product underneath the said two first Figures of the Dividend, Subtracting it from them, and then the Work will stand

Thus 8) 68552 (8
64
4

In order to a second Operation, I make a Point under the next Figure of the Dividend, viz. under the 5, and bring it down underneath its own place to the Remainder 4, which will by that means become 45. Then I consider how many times 8 can be taken from 45, and find it may be 5 times; for 5 times 8 is 40. I therefore place 5 in the Quotient, and with it Multiply 8 the Divisor, setting down and Subtracting their Product, as before. Then the Work will stand

Thus

Thus 8) 68552 (85

64

45

40

5

For a third Operation, I make a Point under the next Figure of the Dividend, viz. under the 5, and bring it down, as before, proceeding in all respects, as before; and then the Work will stand

Thus 8) 68552 (856

64

45

40

55

48

7

Lastly, I point and bring down the 2, viz. the last Figure of the Dividend to the Remainder 7, which will then become 72, and proceeding as in the other Operations, I find that 8 the Divisor can be taken just 9 times from 72, and the Work is finished, and will stand

Thus 8) 68552 (8569

64

45

40

55

48

72

72

(0)

The true Quotient is found to be 8569, being exactly the Eighth part of 68552, or the Multiplicand of the proposed Example of Multiplication. As was required.

The Reason of these Operations will be very plain to any one that will a little consider of it as followeth.

Divisor 8) 68552 (800. The first Quotient or Figure.

Subtract 64000 { This Product of the Divisor into the Quotient is 64000. viz. 8 times 8000. the Quotient Figure being always of the same Value or Degree with that Figure under which the Unit's place of its Product stands.

Divisor 8) 4552 (500. The second Quotient Figure.

Subtract 4000 { And here the Product is 4000. viz. 8 times 500. not 8 times 5. &c.

Divisor 8) 552 (60. The third Quotient Figure.

Subtract 480 { Also here the Product is 480. viz. 8 times 60. for the Reasons abovesaid.

Divisor 8) 72 (9. The fourth Quotient Figure.

Subtract 72 { Now here the Product is but 72. viz. 9 times 8. because the 9 stands in the place of Units.

Remains (00) Now the Sum of all the several Quotients, viz. 8000 + 500 + 60 + 9 = 8569. as before.

If the Process of this Example be well consider'd and compared with that of Multiplication, Page 17. it will evidently appear to be only the Converse of that; for the particular Products are alike in both, only that which is Last there is First here; there they are Added, here they are Subtracted. So that whoever understands the true Reason of the one, must needs understand the Reason of the other, and then Division will become very Easie, although the Divisor consist of several places of Figures.

Example.

Let it be required to Divide 590624922 by 7563

Dividend

Divisor 7563) 590624922 (

'Tis plain at sight, that 7563 the Divisor, cannot be taken from 5906. the like Number of Figures in the Dividend.

Therefore, by the second Case of the Rule (Page 23.) there must be allowed five Figures of the Dividend, viz. 59062 for the first Operation or Quotient; that so the first Figure 7 of the Divisor, may be taken out of the two first Figures, viz. 59 of the Dividend, &c.

E

Then

Then I proceed (*per Case 2.*) and consider how often 7 may be taken from 59. and find it may be taken 8 times, for 8 times 7 is but 56. which I mentally *Subtract* from 59. and there *Remains* 3; to this 3 I mentally adjoyn the *Third Figure* of the *Dividend*, viz. 0. which makes it 30. out of which I must take the *Second Figure* of the *Divisor*, viz. 5. so often as I took the 7 from 59. which was 8 times. But that cannot be, for 8 times 5 is 40. which is more than 30. therefore 8 is too big a *Figure* to be placed in the *Quotient*; Yet, hence I conclude, that the next less, viz. 7 may be taken without any further *Trial*. I therefore place 7 in the *Quotient*, and with it *Multiply* the *Divisor*, setting down their *Product* under the *Dividend*, and *Subtract* it from thence, as in the other *Example*, and then the *Work* will stand

$$\begin{array}{r} \text{Thus } 7563) 590624922 \text{ (7} \\ \underline{52941} \\ 6121 \end{array}$$

In order to a *Second Operation*, I make a *Point* under the next *Figure* of the *Dividend*, viz. under the 4. and bring it down to the *Remainder* 6121. which will then become 61214. with which I proceed in all respects as I did before with the 59062. and find the next *Quotient Figure* will be 8. with which I *Multiply* the *Divisor*, &c. and *Subtract* their *Product* from the said 61214. Then the *Work* will stand

$$\begin{array}{r} \text{Thus } 7563) 590624922 \text{ (78} \\ \underline{52941} \\ 61214 \\ \underline{60504} \\ 710 \end{array}$$

To this *Remainder* 710. I point and bring down the next *Figure* of the *Dividend*, viz. 9. which makes it 7109; now because the *Divisor* 7563 cannot be taken from 7109. I therefore place a *Cypher* in the *Quotient*.

And this must be always carefully observed, viz. That for every *Figure*, or *Cypher*, which is brought down from the *Dividend*, in order to a new *Operation*, there must always be either a *Figure*, or *Cypher* set down in the *Quotient*. Then the *Work* will stand

Thus

Chap. 2. Of Division.

Thus 7563) 590624922 (780

52941

61214

60504

7109

To this 7109, I bring down another *Figure* of the *Dividend*, viz. 2, and then it will become 71192, then I consider how often 7 can be taken from 71 (i.e. just as at the first Operation) and find it may be taken 9 times, therefore I set down 9 in the *Quotient*, and with it *Multiply* the *Divisor*, setting down and *Subtracting* their *Product*, as before; Then the *Work* will stand

Thus 7563) 590624922 (7809

52941

61214

60504

71092

68067

3025

To this *Remainder* 3025, I point and bring down the last *Figure* 2 of the *Dividend*, which makes it 30252, then proceeding in all respects as before, I find the *Quotient Figure* to be 4, with it I *Multiply* the *Divisor*, setting down and *Subtracting* their *Product* as before, and then the *Work* will stand

Thus 7563) 590624922 (78094

52941

61214

60504

71092

68067

30252

30252

(00000)

Here the *Work* is ended, and I find the *Quotient* to be 8094, being the true *Multiplicand* of the propos'd *Example* of *Multiplication*, Page 18.

That is, 7563 is contained in 590624922, just 78094 times, &c.

If the Work of this *Example* be considered and compared with the *Rule* (Page 22) the whole business of *Division* will be easie; for indeed the only difficulty (as I said before) lies in making choice of a true *Quotient Figure*; which cannot well be done according to the Common Method of *Division*, without Trials, yet those Trials need not be made with the whole *Divisor* (as appears by this last *Example*) for by the two first *Figures* of the *Divisor* all the rest are generally regulated; except the second *Figure* chance to be 2, 3, or 4, and at the same time the third *Figure* be 7, 8, or 9, then indeed respect must be had to the third *Figure* according as the *Rule* directs.

However, if those Trials are thought too troublesome, they may be avoided, and the same *Quotient Figures* may both easily and certainly be found by help of such a small Table made of the *Divisor*, as was of the *Multiplicand* in Page 20.

Example 4.

Let it be required to Divide 70251807402 by 79863. See the *Example* of *Multiplication* page 20, and as there directed make a Table of the *Divisor* 79863.

Thus,

Divisor.		Dividend.	
1	79863	70251807402	(879654 Quotient.
2	159726	638904	The work of this Operation I presume may be easily understood. For those <i>Figures</i> in the Table are the <i>Products</i> of the <i>Divisor</i> into all the 9 <i>Figures</i> ; consequently those <i>Figures</i> in the small Column do shew what <i>Figure</i> is to be placed in the <i>Quotient</i> ; without any doubtful Trials of the <i>Divisor</i> with the <i>Dividend</i> as before.
3	239589	636140	
4	319452	559041	
5	399315	770997	
6	479178	718767	
7	559041	522304	
8	638904	479178	
9	718767	431260	
	798630	399315	
		319452	
		319452	
		(000000)	

This Method of Tabulating the *Divisor* may be of good use to a Learner. Especially until he is well practised in *Division*; yea, and even then if the *Divisor* be large, and a *Quotient* of many *Figures* be required, as in Resolving of high *Equations*, and Calculating of *Astronomical Tables*, or those of Interest, &c.

Hitherto

Hitherto I have made choice of *Examples* wherein the *Dividend* is truly Measured or *Divided* off, by the *Divisor*, without leaving any *Remainder*; being those as were composed of the *Divisor* and *Quotient*. But it most usually falls out, that the *Divisor* will not exactly measure the *Dividend*; in that case the *Remainder* (after *Division* is ended) must be set over the *Divisor*, with a small Line betwixt them adjoyning to the *Quotient*.

Example 5.

Suppose it were required to *Divide* 379 by 5.

$$\begin{array}{r} 5 \overline{) 379} \quad (75 \frac{4}{5} \text{ the Remainder,} \\ \underline{35} \quad \text{the Divisor,} \\ 29 \\ \underline{25} \\ \text{Remains) } 4 \end{array}$$

Example 6.

Again, Let it be required to *Divide* 43789, by 67.

67) 43789 (653 $\frac{1}{2}$ The true *Quotient* required

$$\begin{array}{r} 402 \\ \underline{358} \\ 335 \\ \underline{239} \\ 201 \end{array}$$

Remains (38)

How such *Remainders* thus placed over their *Divisors* (which are indeed *Vulgar Fractions*) may be otherwise managed, shall be shew'd farther on.

N. B. When the *Divisor* happens to be an *Unit*, viz. 1 with a *Cypher*, or *Cyphers* annexed to it, As 10, 100, 1000, &c. *Division* is truly performed by cutting off with a *Point* or *Comma*, so many *Figures* of the *Dividend* as there are *Cyphers* in the *Divisor*; then are those *Figures* so cut off to be accounted a *Remainder*, and the rest of the *Figures* in the *Dividend* will be the true *Quotient* required, because *Unit* or 1 doth neither *Multiply* nor *Divide*.

Example 7.

Let it be required to *Divide* 57842 by 100. The Work may stand thus, 100) 578.42 the *Quotient* required, or thus 100) 57842 (578 $\frac{42}{100}$ the same as before.

Hence it follows, that if any *Divisor* have *Cyphers* to the Right hand of it, you may cut off so many of the last *Figures*

in the *Dividend*, and *Divide* the other *Figures* of the *Dividend*, by those *Figures* of the *Divisor* that are left when the *Cyphers* are omitted. But when *Division* is ended those *Cyphers* so omitted in the *Divisor*, and the *Figures* cut off in the *Dividend* are both to be restored to their own places.

Example 8.

Suppose it were required to *Divide* 675469 by 5409.

5400) 675469 (125

542

135

108

274

270

Remains (4) But the true *Remainder* is 469.

Consequently the true *Quotient* is $125\frac{4}{5}\%$

As to the manner of proving the truth of any Operation, either in *Multiplication*, or *Division*, I presume it may be easily understood, by what is deliver'd in *Page 21*, compared with the three first *Examples of Division*; For from thence it will be easie to conceive; that if the *Divisor* and *Quotient* be *Multiplied* together, their *Product* (with what *Remains* after *Division*, added to it) will be equal to the *Dividend*. As in the fifth *Example*, wherein the *Dividend* is 379, the *Divisor* is 5, the *Quotient* is 75 and the *Remainder* is 4.

I say, $75 \times 5 = 375$, to which *Add the Remainder 4*, it will be 379.

Again, in the sixth *Example*, the *Divisor* is 67, the *Quo-*
tient is 653, and the *Remainder* is 38.

Then $653 \times 67 = 43751$, and $43751 + 38 = 43789$ the Dividend, &c.

There are several useful Contractions, both in Division and Multiplication, which I have purposely omitted until I come to treat of Decimal Arithmetick. Also I have omitted the business of Evolution or Extracting of Roots, until further on; and so shall conclude this Chapter with a few Examples of Division unwrought at large, leaving that for the Learner's Practice.

579) 43800771 (75649.

Or, 75649) 43800771 (579

45007)

45007) 23884044718 (530674.

Or 530674) 23884044718 (45007.

356) 244572000 (687000.

59600) 57659066400 (967434.

10000) 679543820000 (67954382.

79) 282016 (3569 $\frac{1}{2}$.

CHAP. III.

Concerning Addition and Substraction of Numbers of different Denominations, and how to Reduce them from one Denomination to another.

SECT. I.

1. Of English Coin.

THE least piece of Money used in England is a Farthing, and from thence ariseth the rest, as in this Table.

Farth.

4 = 1d. Pen.

48 = 12 = 1s. Shil.

960 = 240 = 20 = 1l. Pound Sterling.

And

5 s. is a Crown.

10 s. is an Angel.

6 s. 8 d. a Noble.

13 s. 4 d. a Mark.

Note, When l. s. d. q. are placed over (or to the Right Hand of) Numbers, they denote those Numbers to signify Pounds, Shillings, Pence and Farthings.

l. s. d. q.

As 35 10 6 2. Or 35l. 10s. 6 $\frac{1}{2}$ d. Either of these do signify 35 Pounds, 10 Shillings, 6 Pence, 2 Farthings.

The same must be understood of all the following Characters belonging to their respective Tables, viz Of Weights, Measures, &c.

2. Troy Weight.

The Original of all Weights used in England, was a Corn of Wheat gathered out of the Middle of the Ear, and being well dried, 32 of them were to make one Penny Weight, 20 Pen-

20 Penny Weight one Ounce, and 12 Ounces one Pound Troy. vide Statutes of 51 Hen. 3. 31. Edw. 1. 12 Hen. 7.

But in latter times it was thought sufficient to Divide the aforesaid Penny Weight into 24 equal Parts, called Grains, being the least Weight now in common use; and from thence the rest are computed as in this Table.

Gr. Gra.

24 = 1 P. W. Penny Weight.

480 = 20 = 1 Oz. Ounce.

5760 = 240 = 12 = 1 lb. Pound.

Note,

{ By Troy Weight are weighed Jewels, Gold, Silver, Corn, Bread, and all Liquors.

Besides the common Divisions of Troy Weight, I find in *Anglia Notitia*, or the present State of England, Printed in the Year 1699. that the Moneyers (as that Author calls them) do Subdivide the Grain.

Thus { 24 Blanks = 1 Periot.
20 Periot = 1 Droite.
24 Droites = 1 Mite.
20 Mites = 1 Grain, &c. as before.

3. Apothecaries Weights.

The Apothecaries Divide a Pound Troy, as in this Table.

Gr. Grain.

20 = 1 Scruple.

60 = 3 = 1 Drams.

480 = 24 = 8 = 1 Ounce.

5760 = 288 = 96 = 12 = 1 lb Troy the same as before.

By these Weights the Apothecaries Compound their Medicines, but Buy and Sell their Drugs by *Averdupois Weight*.

4. Averdupois Weight.

When *Averdupois Weight* became first in Use; or by what Law it was first settled, I cannot find out in the Statute Books; but on the contrary, I find that there should be but one Weight (and one Measure) used throughout this Realm, viz. that of Troy, (*Vide* 14. Ed. 3d and 17. Ed. 3.) So that it seems (to me) to be first introduced by Chance and settled by Custom; viz. from giving good or large Weight to those Commodities as are usually weigh'd by it which are such as are either very Course and Drossy, or very

Chap. 3. Of Weights, Measures, &c. 33

ry subject to waft: As, all kind of Grocery Wares, And Pitch, Tar, Rosin, Wax, Tallow, Soap, Flax, Hemp, &c. Copper, Tin, Steel, Iron, Lead, &c. Also Flesh, Butter, Cheese, Salt, &c. To these and the like (I presume) it was thought convenient to allow a greater Weight than what the Laws had provided, which happened to be about a sixth part more: For I found by a very nice Experiment, that one Pound *Averdupois* is equal to 14 Ounces, 11 Penny-weight, and 15 Grains Troy. And it is now computed as in the following Table.

Drams.		lb.	
16	= 1 Oz. Ounces.	14	= a Stone.
256	= 16 = 1 lb. Pounds.	28	= a $\frac{1}{4}$ of C.
28672	= 1792 = 112 = 1 C. Hundred.	56	= $\frac{1}{2}$ a C. or Hun.
573440	= 35840 = 2240 = 20 = 1 Tun.	84	= $\frac{3}{4}$ of C.

5. Long Measure.

As the least part of Weight came at first from a Wheat Corn, so (it's generally said) the least part of Long Measure was at first a Barley Corn taken out of the middle of the Ear, and being well dried three of them in length were to make one Inch; and thence the rest as in this Table.

And $\left\{ \begin{array}{l} 4 \text{ Nails} = \frac{1}{4} \text{ of a Yard.} \\ 1 \frac{1}{4} \text{ Yard} = 1 \text{ Ell.} \\ 2 \text{ Yards} = 1 \text{ Fathom.} \end{array} \right.$

Barley Corns.

3	= 1 In. Inches.		
36	= 12 = 1 F. Feet.		
108	= 36 = 3 = 1 Y. Yards.		
594	= 198 = 16 $\frac{1}{2}$ = 5 $\frac{1}{2}$ = 1 P. Pole or Perch		
23760	= 7920 = 660 = 220 = 40 = 1 Furlong.		
190080	= 63360 = 5280 = 1760 = 320 = 8 = 1 Mile.		

Note, That forty Poles (or Perches) in length, and four in Breadth do make a Statute Acre of Land.

That is, 220 Yards, Multiplied into 22 Yards = 4840 Square Yards is a Statute Acre.

And according to the Transactions of the French Academy Anno 1687, a Paris Foot-Royal is = 128 Inches English; six of those Feet make a Toise; and 57060 Toise = 365184 English Feet, are the Measure of one Degree of a great Circle upon the Surface of the Earth. So that one Degree is 69 Miles, and 288 Yards, which is very near to our Countryman Mr. Normood's Experiment made betwixt London and York; Anno 1635, who found that 367196 Feet = 69 Miles, and

and 858 Yards do make a Degree. And not 60 Miles according to the common received Opinion and Practice of the Navigators or Seamen.

Hence, according to the French account, the Circumference of the Earth (supposing it to be a true Spherical Figure) is, 24899 English Miles.

6. Of Liquid Measures.

All Measures of Capacity, both Liquid and Dry, were at first made from Troy Weight. Vide Statutes 9 H. 3. 51 H. 3. 12. H. 7. &c. wherein it is enacted that eight Pound Troy Weight of Wheat, gathered out of the middle of the Ear and well Dried, should make one Gallon of Wine Measure: And that there should be but one Measure for Wine, Ale and Corn throughout this Realm (Vide Sta. 14 Ed. 3. 15 Ric. 2.) But time and custom hath alter'd Measures, as it hath done Weights (and perhaps for one and the same reason) for now we have three different Measures, viz. one for Wine, one for Ale or Beer, and one for Corn.

I have inserted Tables of each as they are now computed by Cubick Inches, and practised in the Art of Gauging, &c.

The common Wine Gallon sealed at Guild-Hall in London; by which all Wines, Brandies, Spirits, Strong-waters, Meads, Perry, Sider, Vinegar, Oyl and Honey, &c. are Measured and Sold; is supposed to contain 231 Cubick Inches, and from thence the rest are computed, as in this Table.

Cubick Inches.		Galls.
231 = 1 G. Gal ^{ls} .		18 = 1 Runlet, and
9702 = 42 = 1 Terce.	Notes	31½ make a Wine
14553 = 63 = 1½ = 1 Hogshhead.		or Vinegar Barrel.
19404 = 84 = 2 = 1½ = 1 Puncion.		(Vide 1. R. 3.)
29106 = 126 = 3 = 2 = 1½ = 1 Butt or Pipe.		
58212 = 252 = 6 = 4 = 3 = 2 = 1 Tun.		

But Doctor Wybard in his Tectometry, Page 289, doth suppose the Wine-Gallon to contain but 224, or 225 Cubick Inches at the most, and pursuant to his account an Experiment was made by Mr. Walker and Mr. Shales, two General Officers in the Excise, They caused a Vessel to be very exactly made of Brass, in form of a Parallelopipedon, each side of its Base was 4 Inches, and its depth 14 Inches, so that its just content was 224 Cubick Inches. This Vessel was produced at Guild-Hall in London (May 25th 1688) before the Lord Mayor, the Commissioners of Excise, the Reverend

Mr. Flamstead Astr. Reg. Mr. Halley, and several other Ingenious Gentlemen, in whose presence Mr. Shales did exactly fill the foresaid Brazen Vessel with clear Water, and very carefully emptied it into the old Standard Wine-Gallon kept in Guild-Hall, which did so exactly fill it, that all then present were fully satisfied the Wine-Gallon doth contain but 224 Cubick Inches (*this notable Experiment I saw tried.*) However, for several Reasons, it was at that time thought convenient to continue the former supposed content of 231 Cubick Inches to be the Wine-Gallon, and that all Computations in Gauging should be made from thence, as above.

The Beer or Ale-Gallon (which are both one) is much larger than the Wine-Gallon, it being (as I presume) made at first to correspond with Averdupois Weight, as the Wine-Gallon did with Troy-Weight: For (as I said before Page 33) one Pound Averdupois is equal to 14 Ounces 12 Penny Weight Troy.

And, as one Pound Troy is in proportion to the Cubick Inches in a Wine-Gallon, so is one Pound Averdupois: To the Cubick Inches in an Ale-Gallon. That is, 12 : 231 :: 14 $\frac{1}{2}$: 281 $\frac{1}{2}$, very near the Cubick Inches contained in an Ale-Gallon, as appears from an Experiment made by one Nicholas Gunton general Gauger in the Excise, about 36 Years ago, who by such a Vessel mentioned before in the last Page, did find the Standard Ale-Quart (kept in the Exchequer, Vid. 12 Car. 2.) to contain just 70 $\frac{1}{2}$ Cubick Inches, consequently the Ale-Gallon must contain 282 Cubick Inches, and from thence the following Tables are computed.

Ale-Measure.

Cubick Inches.

282	=	1 Gall.
2256	=	8 = 1 Ferkin.
4512	=	16 = 2 = 1 Kilderkin.
9024	=	32 = 4 = 2 = 1 Barrel.
13536	=	48 = 6 = 3 = 1 $\frac{1}{2}$ = 1 Hogshead.

Note, { a Ferkin of Soap, and of Herrings are the same with that of Ale.

Beer-Measure.

Cub. Inches.

282	=	1 Gall.
2538	=	9 = 1 Ferkin.
5076	=	18 = 2 = 1 Kild.
10152	=	36 = 4 = 2 = 1 Barrel.
15228	=	54 = 6 = 3 = 1 $\frac{1}{2}$ = 1 Hogshead.

N. B. This distinction or difference betwixt *Ale* and *Beer-Measure*, is now only used in *London*. But in all other places of *England* the following *Table* of *Beer* or *Ale*, whether it be strong or small, is to be observed, according to a Statute of *Excise* made in the Year 1689.

Cub. Inches.

282	=	1	Gall.
2397	=	8½	= 1 Ferk.
4794	=	17	= 2 = 1 Kild.
9588	=	34	= 4 = 2 = 1 Barrel.
14382	=	51	= 6 = 3 = 1½ = 1 Hoghead.

7. Of Dry Measure.

Dry-Measure is different both from the *Wine* and *Ale-Measure*, being as it were a mean betwixt both, tho' not exactly so; which upon Examination I find to be in proportion to the aforesaid old Standard *Wine-Gallon*, as *Averdupois-Weight* is to *Troy-Weight*; That is, as one *Pound Troy* is to one *Pound Averdupois*, so is the *Cubick Inches* contained in the old *Wine-Gallon*: To the *Cubick Inches* contained in the *Dry* or *Corn-Gallon*.

Viz. $12 : 14\frac{1}{2} : : 224 : 272\frac{1}{2}$ which is very near to $272\frac{1}{4}$ the common received content of a *Corn-Gallon*, altho' now it's otherwise settled by an Act of Parliament made in April 1697, the words of that Act are these:

Every Round Bushel with a plain and even Bottom; being made Eighteen Inches and a half wide throughout, and Eight Inches Deep, should be esteem'd a Legal Winchester Bushel, according to the Standard in his Majesties Exchequer.

Now a Vessel being thus made will contain 2150,42 *Cubick Inches*, consequently the *Corn-Gallon* doth contain but 268½ *Cubical Inches*.

Cub. Inches.

268,8	=	1	Gall.
537,6	=	2	= 1 Peck.
2150,4	=	8	= 4 = 1 Bushel.
7203,2	=	64	= 32 = 8 = 1 Quarter.

Note, { 4 Bushels = a Comb.
10 Quarles = a Wey, and
12 Weys = a Last of Corn.

I observ'd amongst the Lead-Mines in *Derby-shire* (Anno 1692) that the *Miners* bought and sold their Lead Ore, by a Measure which they call'd an Ore-Dish; whose Dimensions I carefully took and found them.

Thus { Length 21.3
Breath 6.
Depth 8.4 } Inches.

Conse-

Consequently it's Content is 1073,52 *Cubick Inches*, which is very near equal to 4, *Corn-Gallons*, according to the above mentioned settlement.

Nine of those Dishes they call a Load of *Ore*, which if it be pretty good, will produce about 3 Hundred Weight of Lead.

8. Of Time.

It is not an easie thing to give a true *Definition* of *Time*; for (according to the *Philosophick Poet*)

'Time of it self is nothing, but from Thought
Receives its Rise, by labouring Fancy wrought
From things consider'd, whilst we think on some
As present, some as past, or yet to come,
No Thought can think on Time, that's still confest,
But thinks on Things in motion, or at rest.

And so on, Vide *Lucretius*, Book 1.

That is, *Time* only shews the *Duration* or *Mutation* of things, a *Year* being the *Standard* or *Integer* by which such Continuance or Change is computed. And a *Year* is that *Space of Time* in which the *Sun* (Apparently) compleats its *Revolution* from any one *Point* in the *Ecliptick* (an imaginary *Circle* in the *Heavens*) to the same *Point* again, which according to *Modern Observations* is perform'd in 365 *Days*, 5 *Hours*, 48 *Minutes*, 57 *Seconds*, 21 *Thirds*, &c. But a *Second* being the least part of *Time* that can be truly Measured by the *Motion* of any *Mechanical Engin*, as a *Clock*, &c. (A third being less than the twinkling of an *Eye*) I begin the following *Table* with *Seconds*.

Seconds.	"	.
60	=	1 Minute.
3600	=	60 = 1 Hour
86400	=	1440 = 24 = 1 Day
31556937	=	525949 = 8765 = 365 + 5 + 48 + 57 = 1 Year,
called a Solar Year.		

But the common *Year*, usually called the *Julian Year*, doth consist of 365 *Days* and 6 *Hours*, and is divided into twelve unequal *Months*, called *Calender Months*, whose Names and Number of *Days* are the Subject of every *Almanack*.

To these *Tables* it may not be amiss to give a brief Account of such *Coins, Weights* and *Measures* as are frequently mention'd in the *Scriptures*. As I have Deduc'd them from those which seem to be the most Correct, inserted in the *Index* to the *Large Bible*, Printed *Anno* 1702, and compared with those used in *England*, by the Lord Bishop of *Peterborough*.

The *Hebrew Weights*, compared with $\left\{ \begin{array}{l} \text{Troy Weight.} \\ \text{Oz. Pw. Grains.} \end{array} \right.$

<i>A Gerah</i>	=	0 . 0 . 10 $\frac{1}{2}$
10 <i>Gerahs</i> = <i>a Bekah</i>	=	0 . 4 . 13 $\frac{1}{2}$
2 <i>Bekahs</i> = <i>a Shekel</i>	=	0 . 9 . 3
100 <i>Shekels</i> = <i>a Maneh</i>	=	45 . 12 . 12

Note, A *Shekel* is said to be their Original Weight.

Their *Coin*. $\left\{ \begin{array}{l} \text{English Coin.} \\ \text{l. s. d.} \end{array} \right.$

<i>A Silver Menah</i>	=	7 . 1 . 5 $\frac{1}{4}$	Weight 60 <i>Shekels</i> .
<i>Talent of Silver</i>	=	357 . 11 . 10 $\frac{1}{2}$	Weight is 300 <i>Shekels</i> .
<i>Talent of Gold</i>	=	5075 . 15 . 7 $\frac{1}{2}$	The same Weight
<i>The Gold Dram</i>	=	1 . 0 . 4	mentioned in <i>Ez. 2. 19</i> .

The *Roman Money* mentioned in the *New Testament*.

A Denarius, or *Silver Penny* = 7 d. 3 *Farthings*.

Asses of Copper = 0 . 3 *Farthings*.

Assarium = 0 . 1 $\frac{1}{2}$ *Farthing*.

Quadrans = 0 : $\frac{3}{4}$ of a *Farthing*.

A Mite = 0 . $\frac{1}{3}$ of a *Farthing*.

Their *Long Measures*, compared with $\left\{ \begin{array}{l} \text{English Measure.} \\ \text{Yar. Feet. Inch. Parts.} \end{array} \right.$

<i>A Fingers Breath</i>	=	0 . 0 . 0,912
4 <i>Fingers</i> = <i>a Hands Breadth</i>	=	0 . 0 . 3,648
2 <i>Hands</i> = <i>the least Span</i>	=	0 . 0 . 7,296
3 <i>Hands Breadth</i> = <i>the longest Span</i>	=	0 . 0 . 10,944
2 <i>Spans</i> = <i>the Longest Cubit</i>	=	0 . 1 . 9,888
4 <i>Cubits</i> = <i>a Fathom</i>	=	2 . 1 . 3,552
6 <i>Cubits</i> = <i>Ezekiel's Reed</i>	=	3 . 1 . 11,328
400 <i>Cubits</i> = <i>a Stadium</i>	=	243 . 0 . 7,2
10 <i>Stadiums</i> = <i>a Mile</i>	=	2432 . 0 . 0
3 <i>Miles</i> = <i>a Parasang</i>	=	7296 . 0 . 0
Which is 4 <i>English Mile</i> and	=	256.

Their

Their Measures of Capacity, compared with $\left\{ \begin{array}{l} \text{English Wine,} \\ \text{Gal. Pints. Inch.} \end{array} \right.$

<i>A Cotyla</i> =	0 . 0 $\frac{1}{2}$	3,037
<i>A Log</i> =	0 . 0 $\frac{1}{2}$	9,83
4 <i>Log's</i> = a <i>Cab</i> =	0 . 3 .	10,458
10 <i>Cotyla's</i> = an <i>Omer</i> =	0 . 6 .	1,5
3 <i>Cab's</i> = a <i>Hin</i> =	1 . 2 .	2,5
2 <i>Hins</i> = a <i>Seah</i> =	2 . 4 .	5,
3 <i>Seah's</i> = an <i>Epha</i> =	7 . 4 .	15.
10 <i>Epha's</i> = a <i>Chomer</i> =	75 . 5 .	5,625

SECT. 2. Addition of Weights, &c.

The foregoing *Tables* being so well understood, as that you can readily tell (without pausing) how many *Units* of any one *Denomination*, do make one of the next *Superior Denomination* (especially in those *Tables* as are most useful for your *Business*) it will then be as easie to *Add*, or *Subtract* them, as to *Add* or *Subtract Whole Numbers*, due care being taken in placing all *Numbers* that are of one *Denomination* exactly underneath each other. That is to say, in *Money* place *Pounds* under *Pounds*, *Shillings* under *Shillings*, *Pence* under *Pence*, &c. Understand the like in *Weights* and *Measures*, &c. According to their several *Denominations*: Then in *Addition* observe this *Rule*.

Rule. Always begin with those *Figures* of the lowest, or least *Denomination*, and *Add* them altogether into one *Sum*, then consider how many of the next *Superior Denomination* are contain'd in that *Sum*, so many *Units* you must carry to the said next *Superior Denomination* to be *Added* together with those *Figures* that stand there; and if any thing *Remain* over or above those *Units* so carried, that *Overplus* must be set down underneath its own *Denomination*: And so proceed on from one *Denomination* to another untill all be finished.

Example in Coin.

Let it be required to *Add* 35 *l.* 14 *s.* 06 *d.* and 27 *l.* 02 *s.* 10 *d.* and 54 *l.* 13 *s.* 04 *d.* and 10 *l.* 17. 09 *d.* into one *Sum*. These particular *Sums* being placed, as before directed, will stand as in the *Margent*.

Then according to the *Rule*, I begin with the *Pence* (being here the lowest or least *Denomination*) and *Adding* them altogether, I find their *Sum* to be 29 *d.* that is 2 *s.* and 5 *Pence*; (for

(for 24 d. = 2 s. and $29 - 24 = 5$) the 5 d. I set down underneath its own *Denomination*, and carry the 2 s. to the place of *Shillings*, Adding them and all the *Shillings* together, I find the *Sum* to be 48 s. viz. 2 l. 8 s. I set down the 8 s. underneath its own Place of *Shillings*, and carry the 2 l. to the Place of *Pounds*, Adding them and all the *Pounds* together, I find their *Sum* is 128 l. consequently the *Total Sum* required is 128 l. 08 s. 05 d.

Now, for as much as it often happens in keeping Books of *Accompts* (and in other Business) that it is required to Add up large *Sums* of Money, consisting of 30, 40, or more several particular *Sums*, nay, perhaps filling up the whole length of a Sheet of Paper, I humbly conceive in those Cases the best and easiest way will be to part them into *Parcels*, not exceeding above 10 or 12 particular *Sums* in each *Parcel*, and Sum up (on a by Paper) the Particulars in each *Parcel*; that done, Add together all the *Sums* of those *Parcels* into one *Sum*, and that will be the *Total Sum* required.

Also to avoid the making of *Points*, or other *Marks* amongst your *Figures*, it will be convenient to get the following *Tables* by heart.

The Pence-Table.				The Shillings-Table.			
d.	s.	d.	s.	s.	l.	s.	l.
12 = 1.		72 = 6.		20 = 1.		120 = 6.	
24 = 2.		84 = 7.		40 = 2.		140 = 7.	
36 = 3.		96 = 8.		60 = 3.		160 = 8.	
48 = 4.		108 = 9.		80 = 4.		180 = 9.	
60 = 5.		120 = 10.		100 = 5.		200 = 10.	

The Use of these *Tables* is so obvious, that I presume 'tis needless to explain them.

Examples in Addition of Weights.

Troy Weight.				Averdupois Weight.			
lb.	oz.	pw.	Gr.	tn.	c.	q.	lb. oz.
3	09	00	10	12	15	2	24 12
5	08	15	21	7	10	3	21 15
10	10	12	22	0	18	1	14 11
0	11	19	23	1	19	3	27 15
Sum	21	4	09 04	Sum	23	05 0 05 05	

Exam.

Examples in Addition of Long-Measure.

<i>Tards.</i>	<i>qrs.</i>	<i>Nails.</i>	<i>Miles.</i>	<i>Fur.</i>	<i>Poles.</i>	<i>Tards.</i>	<i>Feet.</i>	<i>Inch.</i>
35	2	3	2	6	32	4	2	9
17	3	1	0	7	27	3	1	10
129	1	2	1	3	39	1	2	11
182	3	2	Sum	5	2	19	5	0

I think it needless to set down more *Examples* of this kind, for if these 5 (especially the last) be well understood they will be sufficient to shew how any other may be performed.

Sect. 3. Subtraction of Weights, &c.

Subtraction is but the *Converse* of the precedent Work, and may be performed by observing this *Rule*.

Rule. Begin with the lowest or least Denomination (as before in Addition) and take or Subtract the Figure (or Figures) in that place of the Subtrahend, from the Figure (or Figures) that stand over them of the same Denomination; setting down the Remainder (as in Page 12.) But if that cannot be done, then you must increase the upper Figure (or Figures) with one of the next Superior Denomination, and from that Sum make Subtraction; and so proceed to the next Superior Denomination, where you must pay the one borrowed, by adding Unity to the Subtrahend in that place, &c. as in whole Numbers.

Example in Coin.

	<i>l.</i>	<i>s.</i>	<i>d.</i>		<i>l.</i>	<i>s.</i>	<i>d.</i>
From	386	09	08	From	569	10	06
Take	173	04	06	Sub.	389	15	08
Remains	213	05	02		179	14	10

The first of these *Examples* is self evident. In the second *Example*, beginning at the place of *Pence* (being here the least Denomination) I am to take 8*d.* from 6*d.* but because that cannot be done, I must (according to the *Rule*) borrow one of the next Denomination, viz. 1*s.* and Add it to the 6*d.* which makes it 18*d.* (for 1*s.* = 12*d.* and 12*d.* + 6*d.* = 18*d.*) then I take 8*d.* from that 18*d.* and there Remains 10*d.* to be set down underneath the place of *Pence*, that done, I proceed to the place of *Shillings*, where I must now pay the 1*s.* saying one borrowed and 15 makes

makes 16 from 10 cannot be, but 16 from 30 and there *Remains* 14. That is, I borrow one of the next *Denomination*, viz. 1 l. and add it to the 10 s. which makes it 30 s. (for 1 l. = 20 s. and $20 + 10 = 30$) having set down the *Remaining* 14 s. underneath its own place of *Shillings*, I proceed to the place of *Pounds*, where paying the 1 l. borrowed, it will be 1 borrowed and 9 is 10 from 9 cannot be, but 10 from 19 and there *Remains* 9, and so on as in whole *Numbers* until all be finished; And the *Remainder* will be 179 l. 14 s. 10 d.

This *Example* being a little considered will render all others in this *Rule* easie.

Examples in Weights.

Troy-Weight.					Averdupois-Weight.				
	lb.	oz.	pwt.	gr.		c.	qr.	lb.	oz.
From	9	10	16	18		17	2	15	10
Take	5	09	18	22		14	3	18	12
	4	00	17	20	Rests	2	2	24	14

Examples in Long-Measure.

	yards.	qrs.	nails.	miles.	fur.	pot.	yards.	feet.	inches.
From	78	3	2	22	3	26	3	1	11
Take	29	3	3	18	6	29	4	2	7
Rests	48	3	3	3	4	36	4	0	10

Example in Time.

	days.	"	"	"
From	27	18	35	21
Subtract	16	21	46	36
Remains	10	20	48	45

The Proof of *Addition* and *Subtraction* in these *Numbers* of different *Denominations*, is the very same with that of whole *Numbers* in Page 13. I shall therefore refer you to that place, and omit repeating it here.

Sect. 4. Of Reduction.

By *Reduction*, *Numbers* of different *Denominations* are brought into one *Denomination*.

That is, it alters or changes any Superior *Denomination* proposed, into any Inferior or Lesser *Denomination* required.

still keeping them equivalent in value. And by that means they become fitly prepared for *Multiplication* and *Division*; which otherwise could not so conveniently be performed. Therefore the business of *Reduction* is very useful in the *Rule of Proportion* (commonly called the *Golden Rule*, or *Rule of three*) especially to those who do not understand either *Vulgar* or *Decimal Fractions*: And it's thus performed:

Rule. Consider how many Units of the Denomination required, makes one of that Denomination proposed to be Reduced (which is easily known by its respective Table) and with those Number of Units, Multiply the Denomination proposed, and their Product will be the Number required.

Example in Coin.

Let it be required to Reduce or Change 357 *l.* into *Shillings*, and those *Shillings* into *Pence*, which shall still be equal in value with the 357 *l.*

Multiply with $\begin{array}{r} 357 \\ 20 \end{array}$ the *Shillings* in one *Pound*.

Multiply with $\begin{array}{r} 7140 \\ 12 \end{array}$ the *Pence* in one *Shilling*.

$\begin{array}{r} 1428 \\ 714 \end{array}$

$\begin{array}{r} 714 \\ 85680 \end{array}$

$\begin{array}{r} 85680 \end{array}$ = the *Pence* in 357 *l.* as was required.

Or 357 *l.* may be Reduced into *Pence*, at one Operation. Thus,

Multiply with $\begin{array}{r} 357 \text{ l.} \\ 240 \end{array}$ the *Pence* contain'd in one *Pound*.

$\begin{array}{r} 1428 \\ 714 \end{array}$

$\begin{array}{r} 714 \\ 85680 \end{array}$

$\begin{array}{r} 85680 \end{array}$ = the *Pence* in 357 *l.* as before.

But when the *Numbers* proposed to be Reduc'd are of several *Denominations*, and it is required to bring them all to the lowest; you must Reduce the highest or greatest *Denomination* to the next less, Adding the *Numbers* that are of that less *Denomination* together, then Reduce their *Sum* to the next lower *Denomination*, Adding together all the *Numbers* that are of that *Denomination*, and so proceed gradually on until all is done.

Example.

Let it be required to Reduce 375 l. 17 s. 10 d. 3 q. into one Denomination. viz. into Farthings.

$$375 \text{ l. } 17 \text{ s. } 10 \text{ d. } 3 \text{ q.}$$

$$\underline{20}$$

$$7500 = \text{the Shillings in } 375 \text{ l.}$$

$$+ 17 \text{ s.}$$

$$7517 = \text{the Shillings in } 375 \text{ l. } 17 \text{ s.}$$

$$\underline{12}$$

$$15034$$

$$7517$$

$$90204 = \text{the Pence in } 375 \text{ l. } 17 \text{ s.}$$

$$+ 10 \text{ d.}$$

$$90214 = \text{the Pence in } 375 \text{ l. } 17 \text{ s. } 10 \text{ d.}$$

$$\underline{4}$$

$$360856 = \text{the Farthings in } 375 \text{ l. } 17 \text{ s. } 10 \text{ d.}$$

$$+ 3 \text{ q.}$$

$$360859 \text{ Farthings} = 375 \text{ l. } 17 \text{ s. } 10 \text{ d. } 3 \text{ q. as was required.}$$

The work of this *Example*, and all other Operations of this kind, may be somewhat shortned by observing the following Method.

$$375 \text{ l. } 17 \text{ s. } 10 \text{ d. } 3 \text{ q.}$$

$$\underline{20}$$

Multiply and Add in the 17 s.

$$7517$$

$$\underline{12}$$

Multiply and Add in the 10 d.

$$15034$$

$$7517$$

$$90214$$

$$\underline{4}$$

Multiply and Add in the 3 qrs.

$$360859$$

the Farthings as before.

Example in Troy-Weight.

Suppose it be required to Reduce 20 lb. 8 oz. 18 pwt. 21 gra. into the least Denomination, viz. into Grains.

Thus,

Multiply with 12 the oz. in 1 lb. and Add in the 8 oz.

66

29

356 = the Ounces in 29 lb. 8 oz.

Multiply with 20 the pwt. in 1 oz. and Add in the 18 pwt.

7138 = the pwt. in 29 lb. 8 oz. 18 pwt.

Multiply with 24 the Grains in 1 pwt. and Add in the 21 gr.

28553

14278

171333 the Grains = 29 lb. 8 oz. 18 pwt. 21 gr.

These two Examples at large being well understood, may suffice to shew how all Operations of this kind are perform'd; either in *Weights, Measures, or Time*. I shall only insert a few Examples of each sort for the Learners practice.

1. In 23 C. 3 gr. 21 lb. 9 oz. *Averdupois Weight*; How many Ounces. Answer 42905 Ounces.

2. In 252 *English Miles*, How many Yards; Feet and Inches. Answer 443520 Yards = 1330560 Feet = 15966720 Inches.

3. In 1692 common Years, How many Days, Hours and Minuts. Answer 618003 Days, 14832072 Hours, 889924320 Minuts.

Note, a common Year = 365 Days, 6 Hours, see Page 37.

4. In 5786 Pounds, 17 Shill. 9 Pence Sterling: How many Shillings, Pence and Farthings.

Answer, 115737 s. 1388853 d. or 5555412 Farthings. That is, 5786l. 17s. 9d. = 115737 s. 9d. = 1388853 d. &c.

The next thing will be to shew how to bring Numbers from a lesser to a greater Denomination, which by most Authors is called (tho' very improperly)

Reduction Ascending.

This work is the converse of the last, and is perform'd by Division. Thus,

Rule. Consider how many of the Denomination proposed, makes one of the Denomination required, and make that Number your Divisor, by which Divide the Denomination proposed; and the Quotient will be the Number required.

Example.

Thus 16) $\overset{28)}{42905}$ (2681 lb. $\overset{4)}{(95 \text{ qrs.}}$ (23 C.
 $\begin{array}{r} 109 \\ 130 \\ 25 \\ (9) \end{array}$ $\begin{array}{r} 252 \\ 161 \\ 140 \\ (21) \end{array}$ $\begin{array}{r} 15 \\ (3) \end{array}$
 Answer 23 C. 3 qr. 21 lb. 9 oz.

2. In 15966720 Inches; How many English Miles, &c.

Answer 252 Miles. &c. As occasion requires.

There are many useful Questions may be answered by help of Reduction only; As the changing of one sort of Coin for another; and comparing of one sort of Measure with another, &c.

For Instance; Suppose one had 347 Rex Dollers, at 4 s. 6 d. per Doller; and desired to know how many Pounds Sterling they make.

$\begin{array}{r} 347 \\ 54 = \text{the Pence in one Doller, viz. } 4 \text{ s. } 6 \text{ d.} = 54 \text{ d.} \\ 1388 \\ 1735 \end{array}$ 20)
 12) $\begin{array}{r} 18738 \text{ d.} \\ 67 \\ 73 \\ 18 \\ (6) \end{array}$ (1561 s. (78 l.
 $\begin{array}{r} 161 \\ (1) \end{array}$

Answer 78 l. 1 s. 6 d. Sterling are = 347. Rex Dollers.

Quest. 2. In 645 Flemish Ells; How many Ells English.

Note, 3 Quarters of a Yard English makes one Ell Flemish, and $1\frac{1}{4}$ or 5 Quarters of a Yard is an English Ell.

Therefore 645

$\begin{array}{r} 3 = \text{the qrs. of a Yard in 1 Ell Flemish.} \\ \text{qrs. in one Ell} = 5 \end{array}$ $\begin{array}{r} 1935 \\ (387 \text{ English Ells for the Answer.} \end{array}$

Quest. 3. Suppose a Bill of Exchange were accepted at London, for the payment of 400 l. Sterling; for the value deliver'd at Amsterdam in Flemish Money at 1 l. 13 s. 6 d. for one Pound Sterling. How much Flemish Money was delivered at Amsterdam.

First, 1 l. 13 s. 6 d. = 402 d. the value of one l. Sterling at Amsterdam.

Then, $402 \text{ d.} \times 400 = 160800 \text{ d.} = 670 \text{ l. Flemish, and so much was deliver'd at Amsterdam.}$

C H A P.

C H A P. IV.

Of Vulgar Fractions.

S E C T. I. Of Notation.

A *Fraction* or *Broken Number*, is that which represents a *Part* or *Parts* of any thing proposed (*vide* Page 3.) and is expressed by two *Numbers* placed one above the other with a *Line* drawn betwixt them.

Thus, $\left\{ \begin{array}{l} 3 \text{ Numerator.} \\ 4 \text{ Denominator.} \end{array} \right.$

The *Denominator* or *Number* placed underneath the *Line*, denotes how many *Equal Parts* the thing is supposed to be *Divided* into (being only the *Divisor* in *Division*.) And the *Numerator* or *Number* placed above the *Line*, shews how many of those *Parts* are contained in the *Fraction* (it being the *Remainder* after *Division*.) (*See* Page 29.) And these admit of three *Distinctions*.

Viz. $\left\{ \begin{array}{l} \text{Proper or Simple} \\ \text{Improper} \\ \text{Compound} \end{array} \right\} \text{Fractions.}$

A *proper*, *pure*, or *Simple Fraction*, is that which is less than an *Unit*. That is, it represents the immediate *Part* or *Parts* of any thing less than the whole, and therefore its *Numerator* is always less than the *Denominator*.

As $\left\{ \begin{array}{l} \frac{1}{4} \text{ is one Fourth Part.} \\ \frac{1}{3} \text{ is one Third Part.} \end{array} \right.$ And $\left\{ \begin{array}{l} \frac{1}{2} \text{ is one Half.} \\ \frac{2}{3} \text{ is two Thirds. \&c.} \end{array} \right.$

An *improper Fraction*, is that which is greater than an *Unit*; That is, it represents some *Number* of *Parts* greater than the whole thing; And its *Numerator* is always greater than the *Denominator*.

As $\frac{3}{2}$ Or $\frac{5}{2}$ Or $\frac{4}{3}$ &c.

A *Compound Fraction* is a *Part* of a *Part*, consisting of several *Numerators* and *Denominators* connected together with the word (*of*.)

As $\frac{1}{3}$ of $\frac{2}{4}$ of $\frac{3}{5}$, &c. and are thus read, the one Third of the three *Fourths*, of the two *Fifths* of an *Unit*.

That is, when a *Unit* (or whole thing) is first *Divided* into any *Number* of *Equal Parts*, and each of those *Parts* are *Subdivided* into other *Parts*, and so on: Then those last *Parts*

Parts are called *Compound Fractions*, or *Fractions of Fractions*.

As for instance, suppose a *Pound Sterling* (or 20 s.) be the *Unit* or *Whole*; then is 8 s. the $\frac{2}{5}$ of it, and 6 s. the $\frac{1}{4}$ of those two *Fifths*, and 2 s. is the $\frac{1}{3}$ of those three *Fourths*. *Viz.* 2 s. = $\frac{1}{3}$ of $\frac{1}{4}$ of $\frac{2}{5}$ of one *Pound Sterling*.

All *Compound Fractions* are reduced to single ones Thus,
Rule. Multiply all the Numerators into one another for a Numerator, and all the Denominators into one another for the Denominator.

Thus the $\frac{1}{3}$ of $\frac{1}{4}$ of $\frac{2}{5}$ will become $\frac{2}{60}$. Or $\frac{1}{30}$. For $1 \times 3 \times 2 = 6$ the Numerator, and $3 \times 4 \times 5 = 60$ the Denominator, but $\frac{2}{60}$ or $\frac{1}{30}$ of a *Pound Sterling* is 2 s. As above.

Sect. 2. To Alter or Change different Fractions into one Denomination retaining the same Value.

In order to gain a clear understanding of this *Section*, it will be convenient to premise this *Proposition*, *viz.* If a Number Multiplying two Numbers, produce other Numbers, the Numbers Produced of them shall be in the same Proportion, that the Numbers Multiplied are, 17 *Euclid* 7.

That is to say, If both the Numerator and Denominator of any Fraction be equally Multiplied into any Number, their Products will retain the same value with that Fraction.

As in these, $\frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$. Or $\frac{2}{3} \times \frac{3}{3} = \frac{6}{9}$. Or $\frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$ &c.

That is, $\frac{2}{3}$ and $\frac{4}{6}$. Or $\frac{2}{3}$ and $\frac{6}{9}$. Or $\frac{2}{3}$ and $\frac{8}{12}$ are of the same value in respect to the whole or *Unit*.

From hence it will be easie to conceive how two, or more Fractions that are of different Denominations, may be alter'd or chang'd into others that shall have one common Denominator, and still retain the same value.

Example.

Let it be required to change $\frac{2}{3}$ and $\frac{1}{4}$ into two other Fractions that shall have one common Denominator, and yet retain the same value.

According to the foregoing Proposition, if $\frac{2}{3}$ be equally Multiplied with 7, it will become $\frac{14}{21}$. *viz.* $\frac{2}{3} \times \frac{7}{7} = \frac{14}{21}$. Again if $\frac{1}{4}$ be equally multiplied with 3, it will become $\frac{3}{12}$. *viz.* $\frac{1}{4} \times \frac{3}{3} = \frac{3}{12}$. And by this means I have obtained two new Fractions $\frac{14}{21}$ and $\frac{3}{12}$ that are of one Denomination and the same value with the two first propos'd, *viz.* $\frac{14}{21} = \frac{2}{3}$ and $\frac{3}{12} = \frac{1}{4}$.

H

Also

And from hence doth arise the General Rule for bringing all *Fractions* into one *Denomination*.

Rule. *Multiply all the Denominators into each other for a new (and common) Denominator. And each Numerator into all the Denominators but its own, for new Numerators.*

Example.

Let the proposed *Fractions* be, $\frac{1}{3}$. $\frac{2}{5}$. $\frac{3}{4}$ and $\frac{6}{7}$.

Then by the *Rule*.

A new *Denominator* will be thus found.

And the new *Numerators* will be thus found.

3	1	2	3	6
$\frac{5}{15}$	$\frac{5}{5}$	$\frac{3}{6}$	$\frac{3}{9}$	$\frac{3}{18}$
$\frac{4}{60}$	$\frac{4}{20}$	$\frac{4}{24}$	$\frac{5}{45}$	$\frac{5}{90}$
$\frac{7}{420}$	$\frac{7}{140}$	$\frac{7}{168}$	$\frac{7}{315}$	$\frac{4}{360}$

Hence 420 is the common *Denominator*; And 140 . 168 . 315 . 360 are the new *Numerators*, which being placed *Fraction-wise* are $\frac{140}{420}$. $\frac{168}{420}$. $\frac{315}{420}$. $\frac{360}{420}$ the new *Fractions* required.

That is, $\frac{140}{420} = \frac{1}{3}$. $\frac{168}{420} = \frac{2}{5}$. $\frac{315}{420} = \frac{3}{4}$. and $\frac{360}{420} = \frac{6}{7}$.

Sett. 3. To bring mix'd *Numbers* into *Fractions* and the contrary.

Mix'd *Numbers* are brought into improper *Fractions* by the following *Rule*.

Rule. *Multiply the Integers or whole Numbers, with the Denominator of the given Fraction, and to their Product Add the Numerator, the Sum will be the Numerator of the Fraction required.*

Example.

$9\frac{2}{3}$ by the *Rule* will become $\frac{44}{3}$. For $9 \times 3 = 27$. And $27 + 2 = 29$ the improper *Fraction* required.

Again, $13\frac{1}{3}$ will become $\frac{40}{3}$. For $13 \times 3 = 39$ and $39 + 1 = 40$. And so for any other as occasion requires.

To find the true value of any improper *Fraction* given is only the converse of this *Rule*. For if $\frac{44}{3} = 9\frac{2}{3}$ as before is evident.

evident: Then it follows that if 49 be Divided by 5, the Quotient will give $9\frac{4}{5}$. And if 206 be Divided by 15 it will give $13\frac{1}{3}$ &c. consequently it follows, That

If the Numerator of any improper Fraction, be Divided by its Denominator, the Quotient will discover the true value of that Fraction.

Examples.

$1\frac{1}{2} = 5$. And $4\frac{1}{2} = 4\frac{1}{2}$. And $6\frac{1}{2} = 6\frac{1}{2}$. Or $3\frac{1}{4} = 3\frac{1}{4}$ &c.

When whole Numbers are to be express'd Fraction-wise, it is but giving them an Unit for a Denominator. Thus 45 is $45\frac{1}{1}$. 9 is $9\frac{1}{1}$. and 25 is $25\frac{1}{1}$ &c.

Sect. 4. To Abbreviate or Reduce Fractions into their lowest or least Denomination.

This is done, not out of any necessity, but for the more convenient managing of such Fractions as are either proposed in large Terms; or swell into such, either by Addition or otherwise: Besides it's most like an Artift to express or set down all Fractions in the lowest Terms possible; And to perform that, it will be necessary to consider of these following Propositions,

Numbers are either Prime or Composed.

1. A Prime Number is that which can only be Measured by an Unit. Euclid 7. Defi. 11.

That is, 3. 5. 7. 11. 13. 17. &c. are said to be Prime Numbers, because 'tis not possible to Divide them into equal parts by any other Number but Unity or 1.

2. Numbers Prime the one to the other, are such as only an Unit doth Measure, being their common Measure. Euclid 7. Defi. 12.

For instance, 7 and 13 are Prime Numbers to each other, because they cannot be Divided by any Number but an Unit. And 9 and 14 are also Prime Numbers to each other, for altho' 3 will Measure or Divide 9 without leaving a Remainder, yet 3 will not Measure 14 without leaving a Remainder; Again, altho' 2 will Measure 14 without any Remainder, yet 2 will not Measure 9 without leaving a Remainder, &c.

3. A Composed Number is that which some certain Number Measureth. Euclid. 7. Defi. 13.

For instance, 15 is a composed Number of 3 and 5 for $3 \times 5 = 15$, consequently 3 or 5 will justly Measure 15. Also 20

is composed of 5 and 4, viz. $5 \times 4 = 20$, therefore 5 and 4 will each justly Measure 20.

4. Numbers composed the one to the other, are they which some Number being a common Measure to them both doth Measure. Euclid 7. Defi. 14.

That is, If two or more Numbers, can be Divided by one and the same Divisor; then are those Numbers said to be composed one to another.

For instance, 14 and 21 are Numbers composed the one to the other, because they can both be Measured or Divided by 7. For $7 \times 2 = 14$, and $7 \times 3 = 21$, therefore 7 is a common Measure to 14 and 21. So that if $\frac{14}{21}$ were proposed to be abbreviated, it will become $\frac{2}{3}$.

$$\text{Thus } \left\{ \begin{array}{l} 7 \mid 14 = 2 \\ 7 \mid 21 = 3 \end{array} \right.$$

And how those greatest common Measures may be found comes from Euclid 7. pro. 1. 2. 3. and is thus:

Rule. Divide the greater Number by the lesser, and that Divisor by the Remainder (if there be any) and so on continually until there be no Remainder left: Then will that last Divisor be the greatest common Measure (and if it happen to be 1, then are those Numbers Prime Numbers, and are already in their lowest Terms, but if otherwise) Divide the Numbers by that last Divisor, and their Quotients will be their least Terms required.

Example.

Let it be required to find the greatest common Measure of 72 and 108, viz. Of $\frac{72}{108}$.

$$72 \mid 108 \quad (1$$

$$\frac{72}{36}$$

$$36 \mid 72 \quad (2$$

$$\frac{72}{36}$$

$$(0)$$

Here because there's no Remainder 36 is the greatest common Measure.

Therefore $\left\{ \begin{array}{l} 36 \mid 72 = 2 \\ 36 \mid 108 = 3 \end{array} \right.$ Hence $\frac{72}{108}$ is Abbreviated to $\frac{2}{3}$ the lowest Terms.

Again, to find the greatest common Measure of 744 and 899.

Thus

Thus, 744) 899 (1

744

155) 744 (4

620

124) 155 (1

124

31) 124 (4

124

(0)

Here 31 is found to be the greatest common Measure by which 744 and 899 may be Abbreviated to 24 and 29 their lowest Terms.

Thus, $\frac{31}{31}) \frac{744}{899} (= \frac{24}{29}$. &c.

Note, If the Proposed Numbers be even, they may be brought lower by a continued Halving of them, so long as they can be Halved.

Example.

'Tis required to Reduce $\frac{16}{84}$ to its least Terms.

First, $\frac{2}{2}) \frac{16}{84} (= \frac{8}{42}$. Again $\frac{2}{2}) \frac{8}{42} (= \frac{4}{21}$.

This done, you may easily perceive that 7 will be the common Measure to 14 and 21, viz. $\frac{7}{7}) \frac{4}{21} (= \frac{2}{3}$. &c.

If the Numbers proposed to be Reduced have each a Cypher, or Cyphers Annexed to them, they will be Abbreviated by cutting off a like Number of Cyphers from both.

Thus, $\frac{130}{300}$ will be $\frac{13}{30}$. And $\frac{200}{300}$ will be $\frac{2}{3}$. &c.

That is, $\frac{130}{300} = \frac{13}{30} = \frac{1}{2}$. and $\frac{200}{300} = \frac{2}{3}$. also $\frac{360}{400} = \frac{36}{40} = \frac{9}{10} = \frac{2}{10}$

Sect. 5. Addition of Fractions.

What hath been done by the Rules in this Chapter; is chiefly to prepare and fit Fractions of different Denominations for Addition or Subtraction, as Occasion requires. viz. If they are Compound Fractions, they must be Reduced to Simple or Pure Fractions. per Rule, Sect. 1.

If they are of different Denominations, they must be altered or chang'd. per Rule, Sect. 2.

That is, all Fractions must be brought into one Denomination before they can either be Added or Subtracted, and that being done Addition is thus performed.

Rule. Add together all the Numerators, and their Sum will be a New Numerator, under which Subscribe the Common Denominator.

Exam.

Examples in Simple Fractions.

Let it be proposed to Add $\frac{1}{3}$, $\frac{2}{5}$ and $\frac{1}{4}$ together. First, $\frac{1}{3} = \frac{20}{60}$, $\frac{2}{5} = \frac{24}{60}$, and $\frac{1}{4} = \frac{15}{60}$. per Sect. 2. Then $\frac{20}{60} + \frac{24}{60} + \frac{15}{60} = \frac{59}{60}$. the Sum required, which according to Section 3. is $1\frac{59}{60}$. viz. $\frac{59}{60} = 1\frac{59}{60}$.

Examples in Compound Fractions.

Let it be required to Add $\frac{2}{3}$ and $\frac{2}{5}$ of $\frac{1}{4}$ into one Sum. Then $\frac{2}{3}$ of $\frac{1}{4}$ becomes $\frac{2}{12}$ or $\frac{1}{6}$. per Sect. 1. And (per Sect. 2.) $\frac{2}{5}$ and $\frac{1}{6}$ is $\frac{4}{30}$ and $\frac{5}{30}$. viz. $\frac{2}{5} = \frac{4}{10}$ and $\frac{1}{6} = \frac{5}{30}$ but $\frac{4}{10} + \frac{5}{30} = \frac{11}{15}$ the Sum Required, viz. $\frac{2}{3} + \frac{2}{5}$ of $\frac{1}{4} = \frac{11}{15}$.

Examples in Mix'd Numbers.

'Tis required to Add $5\frac{2}{3}$ to $7\frac{1}{4}$. these per Sect. 3. will be $\frac{17}{3}$ and $\frac{11}{4}$. And $\frac{17}{3}$ and $\frac{11}{4}$ will become $\frac{68}{12}$ and $\frac{33}{12}$. per Sect. 2. Then $\frac{68}{12} + \frac{33}{12} = \frac{101}{12}$, and $\frac{101}{12} = 13\frac{5}{12}$. the Sum Required.

Or you may bring only the Fractions to one Denomination, Thus, $5\frac{2}{3}$ and $7\frac{1}{4}$ will become $5\frac{8}{12}$ and $7\frac{3}{12}$.

Then $5\frac{8}{12} + 7\frac{3}{12} = 12\frac{11}{12}$. That is, $13\frac{5}{12}$. As before.

Sect. 6. *Subtraction of Fractions.*

Rule. Subtract one Numerator from the other (according as the Question requires) and their Difference will be a new Numerator, under which Subscribe the Common Denominator, as in Addition.

Example 1.

Let it be required to take $\frac{2}{3}$ out of $\frac{5}{6}$. First $\frac{2}{3}$ and $\frac{5}{6}$. per Sect. 2. will become $\frac{4}{6}$ and $\frac{5}{6}$. then $\frac{5}{6} - \frac{4}{6} = \frac{1}{6}$, that is, $\frac{5}{6} - \frac{2}{3} = \frac{1}{6}$. As was Required.

Example 2.

'Tis Required to Subtract $\frac{2}{5}$ of $\frac{1}{4}$ from $\frac{1}{2}$. First, $\frac{2}{5}$ of $\frac{1}{4} = \frac{1}{10}$. per Sect. 1. Again $\frac{1}{2}$ and $\frac{1}{10}$ will become $\frac{5}{10}$ and $\frac{1}{10}$. per Sect. 2. Then $\frac{5}{10} - \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$.

Example 3.

From $6\frac{1}{8}$ Subtract $3\frac{1}{4}$. First, $6\frac{1}{8} = \frac{49}{8}$. and $3\frac{1}{4} = \frac{13}{4}$. per Rule Sect. 3. Again, $\frac{49}{8} = \frac{245}{40}$, and $\frac{13}{4} = \frac{125}{40}$. per Rule Sect. 2. Then $\frac{245}{40} - \frac{125}{40} = \frac{120}{40} = 3$. Or other-
wise

wise thus: First, $6\frac{1}{8} = 5\frac{5}{8}$, then bring $\frac{2}{8}$ and $\frac{1}{4}\frac{2}{8}$ into one Denomination, viz. $5\frac{5}{8} = 5\frac{11}{8}$ and $3\frac{1}{4}\frac{2}{8} = 3\frac{5}{8}$.

Then $5\frac{11}{8} - 3\frac{5}{8} = 2\frac{6}{8} = 2\frac{3}{4}$. As before.

Example 4.

Let it be required to Subtract $\frac{1}{4}$ of $\frac{2}{3}$ of $\frac{3}{4}$ from 7. First, $\frac{1}{4}$ of $\frac{2}{3}$ of $\frac{3}{4} = \frac{1}{8}$. And $7 = 6\frac{1}{8}$.

Then $6\frac{1}{8} - \frac{1}{8} = 6\frac{0}{8} = 6\frac{1}{3} = 7 - \frac{1}{3}$ of $\frac{2}{3}$ of $\frac{3}{4}$. As was required.

If these few Examples be well understood, the whole Business of Adding and Subtracting Vulgar Fractions will be easie; which is really much more difficult to perform than either Multiplication or Division, as will appear in the next Section.

Sect. 7. Multiplication of Fractions.

In order to perform either Multiplication or Division, you must prepare the Terms to be Multiplied (or Divided) thus; Reduce Compound Fractions to Simple Ones, per Sect. 1. Bring Mixt Numbers into improper Fractions, and express Whole Numbers Fractionwise, per Sect. 3. Also it will be convenient to Abbreviate them to their smallest Terms when it can be done. Then Multiplication may be thus performed.

Rule. Multiply the Numerators one into another for a New Numerator; and the Denominators one into another, for a New Denominator. And in these

Examples.

1. The Product of $\frac{2}{3}$ into $\frac{3}{4} = \frac{6}{12}$. That is, $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$.

2. And the Product of $\frac{2}{3}$ into $\frac{3}{4} = \frac{6}{12}$. Or $\frac{1}{2}$.

3. Again, the Product of $\frac{2}{3}$ into $\frac{3}{4}$ of $\frac{3}{4} = \frac{6}{16}$. Or $\frac{3}{8}$.

For $\frac{2}{3}$ of $\frac{3}{4} = \frac{6}{12}$. Then $\frac{6}{12} \times \frac{3}{4} = \frac{18}{48} = \frac{3}{8}$.

4. Let it be Required to Multiply 6 with $3\frac{2}{3}$. These prepared for the Work will stand thus. $6 \times 3\frac{2}{3}$.

viz. $6 = \frac{6}{1}$ and $3\frac{2}{3} = \frac{11}{3}$. Then $\frac{6}{1} \times \frac{11}{3} = \frac{66}{3}$. or 22 .

Or otherwise thus, $6 \times 3 = 18$. And $\frac{2}{3} \times 6 = \frac{12}{3} = 4$.

Then $18 + 4 = 22$. As before.

5. Let it be required to Multiply $7\frac{1}{2}$ with $5\frac{1}{2}$. First $7\frac{1}{2} = \frac{15}{2}$. and $5\frac{1}{2} = \frac{11}{2}$. Then $\frac{15}{2} \times \frac{11}{2} = \frac{165}{4} = 41\frac{1}{4}$.

Now the Reason of this Rule for Multiplying of Fractions, and consequently of these Operations, and all other performed by it; will be evident from this following.

Viz.

Viz. if $\frac{4}{3}$ be Multiplied With $\frac{1}{2}$ according to the Rule, their Product will be $\frac{4}{6}$: But $\frac{4}{6} = \frac{2}{3}$.
Now $\frac{4}{2} = 2$. and $\frac{1}{3} = \frac{1}{3}$ per Sect. 3. But $4 \times 2 = 8$. ergo &c.

Sect. 8. Division of Fractions.

The Fractions being first prepared as before directed, Division may be thus performed:

Rule. Multiply the Numerator of the Dividend into the Denominator of the Dividing Fraction for a new Numerator: And Multiply the other Numerator and Denominator together for a new Denominator.

Examples.

1. Let $3\frac{6}{7}$ be Divided by $\frac{1}{2}$. *Viz.* $\frac{1}{2}) 3\frac{6}{7} (\frac{4}{10} = \frac{2}{5}$ the Quotient.

That is, according to the Rule $6 \times 7 = 42$ the new Numerator, and $35 \times 3 = 105$, the new Denominator, &c. As above.

2. Let it be required to Divide $\frac{20}{27}$ by $\frac{1}{2}$. *Viz.* $\frac{1}{2}) \frac{20}{27} (\frac{24}{13} = 1\frac{1}{13}$

For $12 \times 20 = 240$ the new Numerator, and $27 \times 5 = 135$ the new Denominator, &c. as before.

3. Suppose it were required to Divide $1\frac{1}{2}$ by $\frac{2}{3}$ of $\frac{1}{7}$.
First, $\frac{2}{3}$ of $\frac{1}{7} = \frac{2}{21}$. Then $\frac{1}{21}) 1\frac{1}{2} (\frac{7}{10} = 1\frac{1}{10}$.

4. Let $20\frac{2}{3}$ be Divided by $3\frac{2}{3}$. *viz.* $\frac{10}{3} \div \frac{11}{3}$ by $\frac{1}{3}$:
For $20\frac{2}{3} = \frac{10}{3}$, and $3\frac{2}{3} = \frac{11}{3}$. Then $\frac{1}{3}) (\frac{10}{11} = 6$ the Quotient.

5. Let it be Required to Divide $40\frac{2}{3}$ by $5\frac{1}{2}$.
First, $40\frac{2}{3} = \frac{120}{3}$, and $5\frac{1}{2} = \frac{11}{2}$. Then $\frac{11}{2}) \frac{120}{3} (\frac{24}{9} = 2\frac{2}{3}$.
But $\frac{120}{3} \div \frac{11}{2} = 7\frac{4}{11}$ the true Quotient Required.

6. Suppose it were Required to Divide 13 by $\frac{1}{2}$.
First, $13 = \frac{13}{1}$. Then $\frac{1}{2}) \frac{13}{1} (\frac{26}{1} = 13$, the Quotient.

7. Again, let it be required to Divide $\frac{1}{2}$ by 6.
Viz. $\frac{1}{2}) \frac{1}{2} (\frac{1}{12}$ for the Quotient Required.

N. B. From hence you may observe, that when any Whole Number is Divided by a Fraction less than Unity or 1, the Quotient will be greater than the Number propos'd to be Divided: But if any Fraction be Divided by a Whole Number, greater than 1. Then the Quotient will be less than the Dividend: As in the two last Examples.

As to the *Reason* (or *Proof*) of this *Rule* for *Dividing Fractions*: 'Tis only the *Converse* to that of *Multiplication*, and will be very evident from this following.

Let $\frac{4}{8}$. be *Divided* by $\frac{1}{2}$. Which according to the *Rule* is thus, $\frac{4}{8} \div \frac{1}{2} = \frac{4}{8} \times \frac{2}{1} = \frac{8}{8} = 1$. The true *Quotient*. Now $\frac{4}{8} = \frac{1}{2}$. And $\frac{1}{2} \div \frac{1}{2} = 1$. per *Seet. 3*. Consequently $\frac{4}{8}$ *Divided* by $\frac{1}{2}$ is but the same with 8 *Divided* by 2. viz. 2) 8 (4. The *Quotient*, as before. I could have inserted *Geometrical Demonstrations*, for both the *Rules* of *Multiplication* and *Division of Fractions*; but supposing the *Learner* purely unacquainted with those kind of *Demonstrations*. I thought these might be more intelligible to him, especially in this place.

CHAP. V.

Of Decimal Fractions.

When, or by whom this *Excellent Invention* of *Decimal Arithmetick*, was first introduced is uncertain, but doubtless its *Improvements*, and the *Perfection* it's now in is owing to latter years.

Sect. 1. Of Notation.

In *Decimal Fractions*, the *Integer* or *Whole Thing* (whether it be *Coin*, *Weight*, *Measure*, or *Time*, &c.) Is supposed to be *Divided* into *Ten Equal Parts*; and every one of those *Ten Parts* to be *Subdivided* into other *Ten Equal Parts*, &c. *ad infinitum*.

The *Integer* being thus *Divided* into 10, 100, 1000, 10000, &c. *Equal Parts*, becomes the *Denominator* to the *Decimal Fraction*.

Thus, $\frac{1}{10}$. $\frac{1}{100}$. $\frac{1}{1000}$. $\frac{1}{10000}$. $\frac{1}{100000}$. &c.

Now these *Denominators* are seldom or never set down but only the *Numerators*; and those are either distinguished, separated from *Whole Numbers* by a *Point* or *Comma*.

Thus, 5,4 is $5\frac{4}{10}$. 0,7 is $\frac{7}{10}$. 35,05 is $35\frac{5}{100}$. &c.

But before we proceed further in *Notation*, it will be convenient for the *Learner* to consider of the following *Table* (taken out of the Learned Mr. *Oughtred's Clavis Mathematica*) which shews the very *Foundation* of *Decimal Fractions*.

Whole

Whole Numbers.										Decimal Parts.												
5	4	3	2	1	0	1	2	3	4	5	6											
						Units Place.							Etc.									
						Tens.							Parts of a Million.									
						Hundreds.							Parts of 100 Thousands.									
						Thousands.							Parts of Ten Thousands.									
						Tens of Thousands.							Parts of a Thousand.									
						Etc.							Parts of a Hundred.									
													Parts of a Ten, or $\frac{1}{10}$.									

By this *Table* it is evident, that as in whole *Numbers* or *Integers*, every *Degree* from the *Units* Place increases towards the left hand by a Tenfold *Proportion*: So in *Decimal Parts* every *Degree* is decreased towards the right hand by the same *Proportion*, viz. by *Tens*.

Therefore these *Decimal Parts* or *Fractions* are really more *Homogeneous* or agreeing with *Whole Numbers* than *Vulgar Fractions*; for indeed all plain *Numbers* are in effect but *Decimal Parts* one to another.

That is, suppose any *Series* of *Equal Numbers*, as 444 &c. The first 4 towards the Left is Ten times the Value of the 4 in the middle, and that 4 in the middle is Ten times the Value of the last 4 to the Right of it, and but the Tenth Part of that 4 on the Left, &c.

Therefore all or any of them may be taken either as *Integers*, or *Parts* of an *Inger*: If *Integers*, then they must be set down without any *Comma* or *Separating Point* betwixt them: thus, 444. But if *Integers*, and one *Part* or *Fraction*, put a *Comma* betwixt them. thus, 44, 4 which signifies 4 *Whole Numbers*, and 4 *Tenths* of an *Unit*: Again, if two *Places* of *Parts* be required, separate them with a *Coma*, thus 4, 44 viz. 4 *Units*, and 44 *Hundred Parts* of an *Unit*, &c.

From hence (duly compared with the Table) it will be easie to conceive that *Decimal Parts* take their *Denominations* from the Place of their last *Figure*.

That is, $\left\{ \begin{array}{l} 55 = 1 \frac{2}{10} \\ 56 = 1 \frac{24}{100} \\ 556 = 1 \frac{24}{100} \end{array} \right\}$ Parts of an Unit, &c.

Cyphers Annex'd to *Decimal Parts* alter not their *Value*. As .50 and .500, or .5000, &c. are each but 5 *Tenths* of an *Unit*. For $\frac{50}{100} = \frac{1}{2}$. And $\frac{500}{1000} = \frac{1}{2}$. Or $\frac{5000}{10000} = \frac{1}{2}$. Per Sect. 4. of the last Chapter.

But Cyphers prefix'd to *Decimal Parts* Decrease their *Value*, by removing them further from the *Comma*.

Thus, $\left\{ \begin{array}{l} .5 = 5 \text{ Tenth Parts.} \\ .05 = 5 \text{ Parts of a Hundred.} \\ .005 = 5 \text{ Parts of a Thousand.} \\ .0005 = 5 \text{ Parts of Ten Thousands, \&c.} \end{array} \right.$

Consequently the true *Value* of all *Decimal Parts* are known by their *Distance* from the *Units Place*; the which being once rightly understood the rest will be easie.

Sect. 2. Addition and Subtraction of Decimals.

In setting down the proposed *Numbers* to be *Added* or *Subtracted*, great care must be taken in placing every *Figure* directly underneath those of the same *Value*, whether they be *Mix'd Numbers*, or *Pure Decimal Parts*, and to perform that you must have a due regard to the *Comma's* or separating *Points*, which ought always to stand in a direct Line one under another; And to the Right-hand of them carefully place the *Decimal Parts* according to their *Respective Values* or *Distances* from *Unit*. Then

Add or Subtract them as if they were all *Whole Numbers*; And from their *Sum* or *Difference*, cut off so many *Decimal Parts* as are the most in any of the given *Numbers*.

Examples in Addition.

Let it be required to find the *Sum* of these following *Numbers*, viz. $34.5 + 65.3 + 128.7 + 95 + 87.8 + 7.9$, which being truly placed, will stand

$$\begin{array}{r} \text{Thus, } \left\{ \begin{array}{l} 34.5 \\ 65.3 \\ 128.7 \\ 95.0 \\ 87.8 \\ 7.9 \\ \hline 419.2 \end{array} \right. \end{array}$$

Their *Sum* required,

Example 2.

Let it be required to find the Sum of 25,854 + 34,578 + 9,076 + 13,907.

$$\begin{array}{r} 25,854 \\ 34,578 \\ 9,076 \\ 13,907 \\ \hline 83,415 \end{array}$$

83,415 The Sum required.

When the *Decimal arts* propos'd to be *Added* (or *Subtracted*) have not the same *Number* of *Places*, you may for convenience of *Operation* supply or fill up the void *Places*, by *Annexing Cyphers*. As in these *Examples*.

Example 3.

$$\begin{array}{r} 45,0700 \\ 50,7580 \\ 123,0057 \\ 74,7020 \\ 24,8000 \\ \hline 318,3357 \end{array}$$

Example 4.

$$\begin{array}{r} 574,678953 \\ 95,796430 \\ 78,054600 \\ 54,789000 \\ 8,900000 \\ \hline \text{Sum } 812,218983 \end{array}$$

Example 5.

$$\begin{array}{r} 0,975642 \\ 5745257 \\ 000598 \\ 800700 \\ 640530 \\ \hline 3,162727 \end{array}$$

Examples in Subtraction.

Let it Required to find the Difference between 45,375 and 74,284.

Example 1.

$$\begin{array}{r} \text{That is, From } 74,284 \\ \text{Take } 45,375 \\ \hline \text{Remains } 28,909 \end{array}$$

Example 2.

$$\begin{array}{r} \text{From } 437,5 \\ \text{Take } 89,657 \\ \hline 347,843 \end{array}$$

Example 3.

$$\begin{array}{r} \text{From } 75,000 \\ \text{Take } 57,875 \\ \hline \text{Remains } 17,125 \end{array}$$

Example 4.

Let it be Required to find the Excess between 562 and 93,5784.

Example 4.

$$\begin{array}{r} \text{That is, From } 562, \\ \text{Take } 93,5784 \\ \hline \text{the Excess. } 468,4216 \end{array}$$

Example 5.

$$\begin{array}{r} \text{From } 345,7578 \\ \text{Take } 157, \\ \hline 188,7578 \end{array}$$

Note, The two last *Examples* are supposed to be supply'd with *Cyphers*, which if actually done would stand thus,

$$\begin{array}{r} 562,0000 \\ 93,5784 \\ \hline \text{Remains } 468,4216 \end{array} \quad \begin{array}{r} 345,7578 \\ 157,0000 \\ \hline \text{As before, } 188,7578 \end{array}$$

Exam

Example 6.

From 0,547893
Take 0,439798
0,108135

Example 7.

From 1,000000
Take 0,997543
0,002457

The Proof of Addition, and Subtraction in Decimals is the same with that of Whole Numbers, Page 13. &c.

Sect. 3. Multiplication of Decimals.

Whether the Factors or Numbers to be Multiplied are pure Decimals, or Mix'd. Multiply them as if they were all Whole Numbers, and for the true Value of their Product observe this Rule.

Rule. Cut off (viz. Separate with a Comma) so many Places of Decimal Parts in the Product, as there are in both the Factors accounted together. As in these.

Example 1.

3,024
2,23
9072
6048
6048
6,74352

Example 2.

32,12
24,3
9636
12848
6424
780,516

The Reason why such a Number of Decimals Parts must be cut off in the Product, may be easily Deduced from these Examples. Thus,

In Example 1. 'Tis evident, that 3 the whole Number in the Multiplicand, being Multiplied with 2, the whole Number in the Multiplier; can produce but 6 (Viz. $3 \times 2 = 6$) So that of necessity all the other Figures in the Product must be Decimal Parts; according as the Rule directs.

Or, the Rule is evident from the Multiplication of whole Numbers only: Thus, suppose 3000 were to be Multiplied with 200, their Product will be 600000; That is, there will be so many Cyphers in the Product, as are either, or both the Factors. (Vide Page 18.) Now if instead of those Cyphers in the Factors, we suppose the like Number of Decimal Parts; Then it follows, that there ought to be the same Number of Decimal Parts in the Product, as there were Cyphers in the Factors.

Again, the Rule may be otherwise made evident from Vulgar Fractions, thus: Let 32,12 be Multiplied with 24,3 and

and their *Product* will be 780,516 as in *Example 2.* above. Now $32,12 = 32, \frac{12}{100}$. and $24,3 = 24, \frac{3}{10}$. which being brought into improper *Fractions* (per *Sect. 3. Page 50*) will become $32, \frac{12}{100} = \frac{3212}{100}$. and $24, \frac{3}{10} = \frac{243}{10}$.

Then $\frac{3212}{100} \times \frac{243}{10} = \frac{780516}{1000}$. per *Sect. 7. Page 55*.

But $\frac{780516}{1000} = 780, \frac{516}{1000}$. viz. 780,516 as before.

Any of these three ways do I presume, sufficiently prove the truth of the abovesaid *Rule*. &c.

Example 3.

$$\begin{array}{r} 78,546 \\ 436 \\ \hline 471276 \\ 235638 \\ 314184 \\ \hline 34246,056 \end{array}$$

Example 4.

$$\begin{array}{r} 5745 \\ ,0675 \\ \hline 28725 \\ 40215 \\ 34470 \\ \hline 387,7875 \end{array}$$

N. B. It sometimes falls out in *Multiplying* parts, with parts that there well not be so many *Figures* in the *Product*, as there ought to be places of *Decimal* parts by the *Rule*: In that case you must supply their defect by prefixing *Cyphers* to the *Product*; as in these *Examples*.

Example 5.

$$\begin{array}{r} ,2365 \\ ,2435 \\ \hline 11825 \\ 7095 \\ 9460 \\ 4730 \\ \hline ,05758775 \end{array}$$

Example 6.

$$\begin{array}{r} ,0347 \\ ,0236 \\ \hline 2082 \\ 1041 \\ 694 \\ \hline ,00081892 \end{array}$$

When any proposed *Number* of *Decimals*, are to be *Multiplying* with 10 . 100 . 1000 . 10000, &c. 'Tis only removing the separating Point in the *Multiplicand*, so many places towards the right-hand, as there are *Cyphers* in the *Multiplier*.

Thus, $,578 \times 10 = 5,78$ and $,578 \times 100 = 57,8$

Again, $,578 \times 1000 = 578$ Or $,578 \times 10000 = 5780$

These

These things being consider'd it will be easie to *Multiply* Decimals, and determin their true *Products*. As in these following *Examples*.

57,056 *Multiplied* into 0,778 will *Produce* 32,978368

7,6543 into 5,4246 will *Produce* 41,52151578

$$0,56879 \times 0,05674 = 0,0322731446$$

$$0,03246 \times 0,02364 = 0,0007673544$$

$$87649 \times 0,03687 = 3231,61863$$

$$9435786 \times 6,57869 = 620,7511100034$$

$$3,141592 \times 52,7438 = 165,6995001296$$

Now it oftentimes happens, that it will be needless to express all the *Figures* of the *Product* at large (especially when the *Factors* have each of them many places of Decimal parts, as in the two last *Examples*) only so many of them as may suffice for the intended design; and yet the *Product* may be as true to so many *Figures* as are retained, as if the *Factors* had been *Multiplied* at large. And such Compendious Contractions are not only of curiosity, but may also be found of great ease and use to the Ingenious Practitioner; especially in *Resolving Adfectèd Equations*, or in calculating of *Trigonometrical Problems* by the *Natural Sines* and *Tangents*, &c. All which may be thus perform'd.

Viz. Set the Units place of the Multiplier directly underneath that Figure of the Multiplicand, whose place you intend to keep in the *Product*; And place all the other Figures of the Multiplier in a quite contrary order to the usual way. Then in *Multiplying* always begin at that Figure of the Multiplicand which stands over the Figure wherewith you are then *Multiplying*, setting down the first Figure of each particular *Product*, directly underneath one another; yet herein you must have a due regard to the increase which would arise out of the two next Figures to the right-hand of that Figure in the Multiplicand which you then begin with.

Example.

Let it be required to *Multiply* 3,141592 with 52,7438 and let there be only four places of Decimal Parts retain'd in the *Product*.

If the proposed *Numbers* were to be *Multiplied* at large they must stand in a direct order as usual. Thus

Thus $\left\{ \begin{array}{l} 3,141592 \\ 52,7438 \end{array} \right\}$ And would produce Ten places of parts, as in the last Example.

But being 'tis required to have only four places of those parts in the *Product*; set them down as above directed and they will stand:

Thus	3,141592	The <i>Multiplicand</i> placed as before.
	834725	The <i>Multiplier</i> in a reverse order.
	1570796	The <i>Product</i> with 5, regard laid to 5 times 2.
	62832	The <i>Product</i> with 2, increased with 9×2 .
	21991	<i>Product</i> with 7, increased with $5 \times 7 + 9 \times 7$.
	1257	<i>Product</i> with 4, increased with $1 \times 4 + 5 \times 4$.
	94	<i>Product</i> with 3, increased with 4×3 .
	25	<i>Product</i> with 8, increased with $4 \times 8 + 1 \times 8$.
	165,6995	The true <i>Product</i> as was required.

The Reason of this Contraction is very obvious from the whole Operation wrought at large.

Thus

3,141592	
52,7438	
25	132736
94	24776
1256	6368
21991	144
62831	84
1570796	0
165,6995	001296

From hence it's evident that all the Figures in the Square to the right-hand, are wholly omitted in the former Contraction; and that the last single *Product* here, is the first there; consequently the Reason of placing the *Multiplier* in a reverse order, must needs appear very plain.

Example 2.

Suppose it were required to Multiply 257,356 with 76,48 and to have only the entire *Product* of Integers.

257,356
84,67
18015
1544
103
20
19682

The same at large.

$\left\{ \begin{array}{l} 257,356 \\ 76,48 \end{array} \right\}$
2058848
1029424
1544136
1801492
19682,58688

The chiefest care and difficulty that attends these Contractions, is the true setting down of the *Units* place in the *Multiplier*, underneath the proper Figure of the *Multiplicand* according to the design'd *Product*.

Viz. In *Example 1.* It was required to have four places of of Decimal Parts in the *Product*; therefore the *Units* place of the *Multiplier* was set under the fourth place of Decimals in the *Multiplicand*: And in *Example 2.*, because it was requir'd to have an entire *Product* of *Integers* only; therefore the *Units* place of the *Multiplier* was set under the *Units* place of the *Multiplicand*. This I say, being once rightly understood, will render the Method easie in Practice.

Sect. 4. Division of Decimals.

Division is accounted the most difficult part of *Decimal Arithmetick*; In order therefore to make it plain and easie, it will be convenient to resume what has been said in Page 25.

Viz. The *Quotient Figure* is always of the same *Value* or *Degree* with that *Figure* of the *Dividend* under which the *Units* place of its *Product* stands.

As for Instance, Let 294 be Divided by 4.

$$\begin{array}{r}
 4) 294 \overset{(7)}{} \\
 \underline{28} \\
 14 \quad (3 \text{ But this is only } 3. \\
 \underline{12}
 \end{array}
 \quad \left. \begin{array}{l} \text{This is not } 7 \text{ but } 70, \text{ because the} \\ \text{Units place of } 4 \times 7 \text{ stands under the} \\ \text{Tens place of the Dividend.} \end{array} \right\}$$

Remains (2) Hence $73\frac{2}{4}$ is the *Quotient*.

Now if to the *Remainder* 2 there be Annexed a *Cypher*, thus 2,0 and then *Divided* on, it must needs follow that the *Units* place of the *Product* arising from the *Divisor* into the *Quotient*, will stand under the Annexed *Cypher*; Consequently the *Quotient Figure* will be of the same value or degree with the place of that *Cypher*: But that's the next below the *Units* place, Therefore the *Quotient Figure* is of the next degree or place below *Unity*; That is, in the first place of *Decimal Parts*.

Thus 4) 2,0 (5

So that 4) 294,0 (73,5 the true *Quotient* required.

This being well understood; *Division* of *Decimals* may in all the various cases) be easily perform'd. However, that it may be render'd plain and easie, even to the meanest capacity if possible; Let *Division* be again defin'd, as

Viz. If that Number which Divides another, be Multiplied with the Number which is produced, their Product will be the Number Divided.

This Definition alone (if compar'd with the Rule Page 61) will afford a General Rule for discovering the true value of the Quotient Figure in Division of Decimals.

Rule. The places of Decimal Parts in the Divisor and Quotient, being accounted together, must always be equal in Number with those in the Dividend. And from this General Rule ariseth four Cases.

Case 1. When the places of parts in the Divisor and Dividend are equal, the Quotient will be whole Numbers. As in these Examples.

$$\begin{array}{r} 8,45 \overline{) 295,75} \quad (35 \\ \underline{253 \ 5} \\ 42 \ 25 \\ \underline{42 \ 25} \\ (0) \end{array}$$

$$\begin{array}{r} 0,0078 \overline{) ,4368} \quad (56 \\ \underline{390} \\ 468 \\ \underline{468} \\ (0) \end{array}$$

Case 2. When the places of parts in the Dividend, exceeds those in the Divisor; cut off the Excess for Decimal Parts in the Quotient. As in these Examples.

$$\begin{array}{r} 24,3 \overline{) 780,516} \quad (32,12 \\ \underline{729} \\ 515 \\ \underline{486} \\ 291 \\ \underline{243} \\ 486 \\ \underline{486} \\ (0) \end{array}$$

$$\begin{array}{r} ,534 \overline{) ,30438} \quad (,57 \\ \underline{2670} \\ 3738 \\ \underline{3738} \\ (0) \end{array}$$

$$\begin{array}{r} 436 \overline{) 34246,056} \quad (78,546 \\ \underline{3052} \\ 3726 \\ \underline{3488} \\ 2380 \\ \underline{2180} \\ 2005 \\ \underline{1744} \\ 2616 \\ \underline{2616} \\ (0) \end{array}$$

Case 3. When there are not so many places of parts in the Dividend, as are in the Divisor; Annex Cyphers to the Dividend to make them equal. Then will the Quotient be whole Numbers, as in Case 1.

Examples.

Let it be required to Divide 192,1 by 7,684, And 441 by ,7875.

$ \begin{array}{r} 7,684 \overline{) 192,100} \quad (25 \\ \underline{153 \ 68} \\ 38 \ 420 \\ \underline{38 \ 420} \\ (0) \end{array} $	$ \begin{array}{r} ,7875 \overline{) 441,0000} \quad (560 \\ \underline{393 \ 75} \\ 47 \ 250 \\ \underline{47 \ 250} \\ (00) \end{array} $
---	--

Case 4. If after Division is finished, there are not so many Figures in the Quotient, as there ought to be places of parts by the General Rule; supply their defect by prefixing Cyphers to it.

Examples.

Let it be required to Divide 7,25406 by 957.

957) 7,25406 (,00758 the true Quotient required.

$ \begin{array}{r} 6 \ 699 \\ \underline{5550} \\ 4785 \\ \underline{7656} \\ 7656 \\ \underline{} \\ (0) \end{array} $	Again ,575) ,0007475 (,0013 $ \begin{array}{r} \underline{575} \\ 1725 \\ \underline{1725} \\ (0) \end{array} $
---	---

Note. When Decimal Numbers are to be Divided by 10 . 100 . 1000 . 10000 &c. That is, when the Divisor is an Unit with Cyphers; Division is performed by Removing or placing the separating point in the Dividend, so many places towards the left hand, as there are Cyphers in the Divisor.

Examples.

$ \begin{array}{l} 10) 5784 \quad (578,4 \\ 1000) 5784 \quad (5,784 \end{array} $	$ \begin{array}{l} 100) 5784 \quad (57,84 \\ 10000) 578,4 \quad (,05784 \end{array} $
--	--

Note. These Operations are the direct converse to those in Page 62.

I presume it needless to give more Examples at large, only insert a few Dividends, and Divisors, with their Quotients; wherein are contained all the varieties that can happen in Division of Decimals.

$ \begin{array}{l} 574) 493066 \quad (859. \\ 574) 493,066 \quad (,859 \\ 574) 49,3066 \quad (,0859 \\ 574) 4930,66 \quad (859 \end{array} $	$ \begin{array}{l} 5,74) 49,3066 \quad (8,59 \\ 5,74) 493066,00 \quad (85900 \\ ,0574) 493,0660 \quad (8590 \\ ,0574) ,463066 \quad (8,59. \end{array} $
---	---

There is also a compendious way of contracting *Division*; Like unto that of *Multiplication* Page 64, by which much labour may be saved; Especially when the *Divisor* hath many places of *Decimal Parts* in it: And its thus performed.

Having determined how many places of whole *Numbers* there will be in the *Quotient*, if any at all; or if none, of what value or place the first *Figure* in the *Quotient* will be: Then omit, or prick off one *Figure* of the *Divisor* at each *Operation*; viz. for every *Figure* you place in the *Quotient*, prick off one in the *Divisor*; having a due regard to the increase which wou'd arise from the *Figure* so omitted.

Example.

Let it be required to *Divide* 70,23 by 7,9863.

The Work contracted.

$$\begin{array}{r}
 7,9863 \overline{) 70,2300} \quad (8,7938 \\
 \underline{63 \ 8904} \\
 6 \ 3396 \\
 \underline{5 \ 5904} \\
 7492 \\
 \underline{7187} \\
 305 \\
 \underline{239} \\
 66 \\
 \underline{64} \\
 (2)
 \end{array}$$

The same at large.

$$\begin{array}{r}
 7,9863 \overline{) 70,2300} \quad (8,7938 \\
 \underline{63 \ 8904} \\
 6 \ 3396 \quad 0 \\
 \underline{5 \ 5904} \quad 1 \\
 7491 \quad 90 \\
 \underline{7187} \quad 67 \\
 304 \quad 230 \\
 \underline{239} \quad 589 \\
 64 \quad 6410 \\
 \underline{63} \quad 8904 \\
 0 \quad 7506
 \end{array}$$

The Work contracted I presume is so obvious (if compared with the same at large) that it's needless to give any farther explanation of it.

Seet. 5. To Reduce Vulgar Fractions into Decimals, and the contrary.

Any *Vulgar Fraction* being given it may be *Reduced*, or rather *Changed* into *Decimal Parts* equivalent to it. Thus, Annex Cyphers to the *Numerator*, and then *Divide* it by the *Denominator*, the *Quotient* will be the *Decimal Parts* equivalent to the given *Fraction*; or at least so near it as may be thought necessary to approach.

Exam.

Example.

'Tis required to *Change* or *Reduce* $\frac{3}{4}$ into *Decimals*.

4) 3,00 (,75 The *Decimal Parts* required,
That is, $\frac{3}{4} = \frac{75}{100} = ,75$.

Again $\frac{1}{2} = ,5$ Thus 2) 1,0 (,5 And $\frac{1}{4} = ,25$ 4) 1,00 (,25
Suppose it were required to *Change* $\frac{3}{7}$ into *Decimals*.

7) 4,0000000000 (,5714285714 &c. = $\frac{3}{7}$

Note. When the last *Figure* of the *Divisor* (That is, the *Denominator* of the proposed *Fraction*) happens to be one of these *Figures*; viz. 1. 3. 7. or 9. (as in the last *Example*) then the *Decimal Parts* can never be precisely equal to the given *Fraction*; yet by continuing the *Division* on, you may bring them to be very near the truth. As in this *Example*; Suppose it required to *Change* $\frac{1}{3}$ into *Decimal Parts*.

13) 1.0000 (,07692307692307 &c. *ad infinitum*.

91

90

78

120

117

30

26

40

39

10

&c.

That is, $0,07692307692307 = \frac{1}{13}$ *ferè*!

And from hence it may be further observed; That in these imperfect *Quotients*, the *Figures* do return again and circulate in the same order as before: As you may easily perceive they begin to do in the seventh place of both these last *Examples*.

As at first.

These being understood, it will be easie to find the *Decimal Parts* equivalent to any known Part, or Parts of *Coin*, *Weights*, *Measures*, or *Time*, &c. If you first *Reduce* the given *Parts* of *Coin*, &c. Into a *Vulgar Fraction*, whose *Denominator* is the *Number* of those known *Parts* contained in the *Integer*, and the given *Parts* its *Numerator*.

Examples in *Coin*, &c.

1. Let it be Required to find the *Decimals* of 16s. 6d. first $16s. = \frac{1}{2} \text{ of one Pound}$, and $6d. = \frac{1}{4} \text{ of 1 l.}$ But $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. Then 40) 33,000 (,825 the *Decimal Parts* Required: That is, $,825 = 16s. 6d.$

Again, Suppose it were Required to find the *Decimals* equal to 3 l. 13s. 4d. Here

Here 3 *l.* is 3 *Integers*, and 13 *s.* = $\frac{1}{2}$ of 1 *l.* and 4 *d.* = $\frac{1}{4}$ of 1 *l.*
 But $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. Then 240) 160,000 (,666666, &c.
 Hence 3 *l.* 13 *s.* 4 *d.* = 3,666666, &c. As was Required.

2. What are the *Decimals* equal to $7\frac{1}{4}$ Inches, one Foot being made the *Integer*.

First, 7 Inches are $\frac{7}{12}$ of 1 Foot, and $\frac{1}{4}$ of 1 Inch are $\frac{1}{48}$.
 But $\frac{7}{12} + \frac{1}{48} = \frac{31}{48}$. Then 48) 31,000 (,64583 &c. = $7\frac{1}{4}$ Inches.

3. Let it be Required to Change 8 $\frac{1}{2}$ 19 Pwt. 8 Grains into *Decimals*; one lb. Troy, being the *Integer*.

These being Reduced into their Least Terms, and Added together will become $\frac{448}{1000}$ of 1 lb.

Then 5760) 4304,000 (,74722, &c. The *Decimals* Required.

And thus may any proposed Parts of Coin, Weights, Measures, &c. Be Reduced or Changed into *Decimal Parts*; which perhaps may at first seem somewhat tedious in Practice, but being a little acquainted with them, it will be found very easie; and the Ingenious Practitioner will (with a little Consideration) soon find how to Reduce them almost mentally; or with the help of a very few Figures; without the use of such large Tables as are usually Inserted in Books of *Decimal Arithmetick*, or at most they may be contracted into such as these following; which if duly applied to those Tables in Chap. 3. Will be found very useful.

Decimal Tables.

<i>In English Coin.</i>	<i>Averdupoise Weight.</i>
0,05 = 1 <i>s.</i>	0,0625 = 1 Ounce
0,00416667 = 1 <i>d.</i>	0,00390625 = 1 Drachm
0,00104167 = 1 Farthing	1, lb. being the Integer
1, <i>l.</i> being the Integer	
<i>Troy Weight.</i>	<i>Averdupoise Great Weight.</i>
0,05 = 1 Pwt.	0,25 = $\frac{1}{4}$ C.
0,00208333 = 1 Grain	0,00892857 = 1 lb.
1, $\frac{1}{3}$. being the Integer	0,00055803 = 1 Ounce
	1, C. being the Integer.
<i>Apothecaries Weights.</i>	<i>Time.</i>
0,125 = 1 Drachm	0,04166667 = 1 Hour
0,04166667 = 1 $\frac{1}{2}$	0,00069444 = 1 Minute
0,00204167 = 1 Grain	0,00001159 = 1 Second
1, $\frac{1}{3}$. being the Integer	1, Day or 24 Hours being made the Integer.

The Use of these Tables will be evident by the following

Exa

Example.

Let it be Required to find the *Decimal Parts* equivalent to 17 s. 9 d. 2 Farthings.

First, $0,05 = 1 \text{ s.}$ Therefore $17 \times 0,05 = 85 \dots = 17 \text{ s.}$

And $,004166 = 1 \text{ d.}$ Therefore $,004166 \times 9 = ,037494 = 9 \text{ d.}$

Also 2) $,004166 (= ,002083 = \frac{1}{2} \text{ d.})$

Consequently their Sum, viz. $0,889577 = 17 \text{ s. } 9 \frac{1}{2} \text{ d.}$

Now to find the *Value* of *Decimals* in known *Parts* of *Coin*, or *Weights*, &c. Is only the *Converse* of the former Work. And is thus performed.

Multiply the given *Decimals* with the *Denominator* of the *Vulgar Fraction* Required: That is, Multiply the *Decimals* with such a *Number* of *Units* as are contained in the next Lower *Denomination* of that *Kind* or *Species* which your *Decimal* is of: And the *Product* will be the *Number* Required.

Example.

1. What is the *Value* of $0,825$ *Decimals* of 1 *Pound Sterling*. That is, how many *Shillings*, *Pence*, &c. $= 825$. First, the next lower *Denomination* is 20, because 20 s. make one *Pound*.

Therefore $0,825$

$\begin{array}{r} 20 \\ \hline \text{Shillings } 16,500 \end{array}$ And *Parts* of 1 *Shilling*.

$\begin{array}{r} 12 \\ \hline \text{Pence } 6,000 \end{array}$ Answer $0,825 = 16 \text{ s. } 6 \text{ d.}$

Again, what are the known *Parts* of *English Coin* equal to $3,666666$ *Decimals*.

Here the 3 *Integers* are 3 *Pounds*. Then $,666666$

$\begin{array}{r} 20 \\ \hline \text{Shillings } 13,333320 \\ \hline 12 \end{array}$

Answer $3,666666 = 3 \text{ l. } 13 \text{ s. } 4 \text{ d.}$

$\begin{array}{r} 666640 \\ 33332 \\ \hline \text{Pence } 3,999840 \end{array}$

What is the *Value* of $0,74722$ *Parts* of 1 *lb. Troy*.

First, $,74722$

Then, $,96664$

Again, $,33280$

$\begin{array}{r} 12 \\ \hline 149444 \\ 74722 \end{array}$

$\begin{array}{r} 20 \\ \hline \text{Pwts. } 19,33280 \end{array}$

$\begin{array}{r} 24 \\ \hline 13312 \\ 6656 \end{array}$

8,96664

3. Pwt. Gr.

7,98720

These Collected are 8. 19. 8. very near.

And

And thus any proposed *Number* of *Decimals* may be turn'd or changed into the known *Parts* of what they Represent. viz. Whether they be *Parts* of *Coin*, *Weights*, *Measures*, or *Time*, &c.

I have omitted inserting more *Examples* of this kind, because I take the Excellency, and indeed the chief use of *Decimal Fractions* to consist more in *Geometrical Computations*, than in the Common or Practical *Parts* of *Arithmetick*, (as will appear further on) although even in those they are very useful upon several Accounts; especially in the *Computations* of *Interest* and *Annuities*, &c. (But of that more in its proper Place.) I shall therefore conclude this Chapter with a Remark or two upon the *Nature* and *Properties* of *Fractions* in General.

If any given *Number* (whether it be *Whole* or *Mix'd*) be *Multiplied* with a *Fraction*, either *Vulgar* or *Decimal*, the *Product* will be less than the *Multiplicand*, in such a Proportion, as the *Multiplying Fraction* is less than a *Unit* or 1.

That is, as the *Denominator* of the *Fraction*; Is to its *Numerator*: So will the given *Number* be; To the *Product*.

Therefore, whenever any *Number* is to be *Multiplied* with a *Fraction*, whose *Numerator* is a *Unit*. Divide that *Number* by the *Denominator* of the *Fraction*, and the *Quotient* will be the *Product* Required. Thus $12 \times \frac{1}{4} = 3$. And $12 \div 4 = 3$. Again $12 \times \frac{1}{2} = 6$. And $12 \div 2 = 6$. &c.

From hence it follows, that if any *Number* be *Divided* by a *Fraction*, the *Quotient* will be greater than the *Dividend*; by such a Proportion as *Unity* is greater than the *Dividing Fraction*.

Thus $12 \div \frac{1}{4} = 48$, viz. $\frac{1}{4} : 1 :: 12 : 48$. &c. But the truth of these will be best understood after the next Chapter.

C H A P. VI.

Of Continued Proportions, and how to Change or Vary the Order of Things.

Sect. I. Concerning Arithmetical Progression, usually called Arithmetical Proportion Continued.

When any Rank or Series of Numbers, do either increase or decrease by an Equal Interval or Common Difference; those Numbers are said to be in Arithmetical Progression.

As $\{ 1. 2. 3. 4. 5. 6. 7. \&c. \}$ $\{ \}$ Here the *Interval* or
 $\{ 7. 6. 5. 4. 3. 2. 1. \}$ $\{ \}$ Common *Difference* is 1.

Or $\{ 2. 4. 6. 8. 10. 12. 14. \&c. \}$ $\{ \}$ Here the Common
 $\{ 1. 3. 5. 7. 9. 11. 13. \&c. \}$ $\{ \}$ *Difference* is 2.

And so of any other Series, whose Common *Difference* is
 $3. 4. 5. \&c.$

Lemma 1.

If any three Numbers be in *Arithmetical Progression*; the
Sum of the two *Extreams* (*viz.* the First and Last) will be
 equal to the Double of the *Mean* or middle Number.

As in these $2 : 4 : 6$. Or $3 : 6 : 9$. Or $3 : 7 : 11$.
Viz. $2 + 6 = 4 + 4$. Or $3 + 9 = 6 + 6$. And $3 + 11 = 7 + 7. \&c.$

Lemma 2.

If any four Numbers are in *Arithmetick Progression*, the
Sum of the two *Extreams* will be equal to the *Sum* of the
 two *Means*.

As in these. $2 : 4 : 6 : 8$. Or $3 : 6 : 9 : 12$.
Viz. $2 + 8 = 4 + 6$. And $3 + 12 = 6 + 9. \&c.$

Corolary 1.

From these two Lemma's it's easie to conceive; that if ne-
 ver so many Numbers be in *Arithmetick Progression*, the
Sum of the two *Extreams* will be equal to the *Sum*
 of any two *Means*; That are equally distant from the *Extreams*.

As in these, $2 : 4 : 6 : 8 : 10 : 12 : 14 : 16$.

Then $2 + 16 = 4 + 14 = 6 + 12 = 8 + 10$.

Or if the Number of Terms be odd as these.

$2 : 4 : 6 : 8 : 10 : 12 : 14 : 16 : 18 : \&c.$

Then $2 + 18 = 4 + 16 = 6 + 14 = 8 + 12 = 10 + 10$.

Lemma 3.

Every Series of Numbers in *Arithmetick Progression* are
 composed of the *Interval* or Common *Difference*, so often re-
 peated as there are Terms in the *Progression* except the first.

As in these $1. 3. 5. 7. 9. 11. 13. 15. 17. \&c.$

Here the *Interval* or Common *Difference* being two, it
 will be $1 + 2 = 3$. $3 + 2 = 5$. $5 + 2 = 7$. $7 + 2 = 9$.
 $9 + 2 = 11$. $11 + 2 = 13$. $13 + 2 = 15$. $15 + 2 = 17. \&c.$

Corolary 2.

Hence it's evident, that the *Difference* betwixt the two
Extreams (*viz.* 1 and 17) is composed of the common *Dif-*
ference, Multiplied into the Number of all the Terms ex-
 cepting the first.

As in the aforesaid *Progression*. $1. 3. 5. 7. 9. 11. 13. 15. 17.$

L

The

The Number of Terms without the First is 8 }
 The Common Difference is 2 } Multiply
 The Difference betwixt the two Extrems 16

Proposition 1.

In any Series of Numbers in Arithmetick Progression, the two Extrems, and the Number of Terms being given; thence to find the Sum of all the Series.

Theorem { Multiply the Sum of the two Extrems into the Number of all the Terms; and Divide the Product by 2. The Quotient will be the Sum of all that Series. Per Corol. 1.

Example 1.

'Tis Required to find the Number of all the Stroaks a Clock strikes in one whole Revolution of the Index. viz. in Twelve Hours.

Here $1 + 12 = 13$ the Sum of the two Extrems.
 And $\frac{12}{26}$ the Number of all the Terms.
 13.

Then $2) 156$ (78. The Number of Stroaks Required.

Example 2.

Suppose one Hundred Eggs were placed in a Right Line, a Yard distant from one another; and the first Egg were a Yard from a Basket; whether may a Man gather up those 100 Eggs singly one after another, still returning with every Egg to the Basket and put it in, before another Man can Run four Miles. That is, which will Run the greater Number of Yards.

In this Question $200 + 2 = 202$ Is the Sum of the two Extrems.
 And $\frac{100}{10100}$ Is the Sum of all the Terms.

Then $2) 20200$ (10100 { The Number of Yards he runs that takes up the Eggs.

Now 4 Miles = 7040 Yards { The Yards he runs that takes up
 But $10100 - 7040 = 3060$ { the Eggs more than the other.

Proposition 2.

In any Series of Numbers in Arithmetick Progression, the two Extrems and Number of Terms being given; thence to find the Common Difference of all the Terms in that Series.

Theorem 2. { The Difference betwixt the two Extrems being Divided by the Number of Terms less an Unit or 1. The Quotient will be the Common Difference of the Series. Per Corol. 2.

Exam

Example 1.

One had Twelve Children that differed alike in all their Ages; the Youngest was Nine Years Old, the Eldest was Thirty Six and a half, what was the *Difference* of their Ages, and the Age of each.

Here $36,5 - 9 = 27,5$. The *Difference* of the two *Extreams*.
 And $12 - 1 = 11$. The *Number* of *Terms* less an *Unit*.
 Then 11) $27,5$ (2,5 The *Common Difference Required*.
 Consequently $9 + 2,5 = 11,5$ The Age of the Youngest but one. And $11,5 + 2,5 = 14$ The Age of the Youngest but two. And so on for the rest. *Per Corol. 2.*

Example 2.

A Debt is to be Discharged at Eleven several Payments to be made in *Arithmetick Progression*. The first Payment to be Twelve Pounds Ten Shillings, and the last to be Sixty Pounds. What's the whole Debt, and what must each Payment be.

Per Theorem 1. Find the whole Debt thus.

$12,5 + 63 = 75,5$ The *Sum* of the *Extreams*.

11 The *Number* of *Terms*.

75 5

755

2) $830,5$ ($415,25 = 415$ l. 5 s. The whole Debt.

Then *Per Theorem 2.* Find the *Common Difference* of each Payment.

Thus $63 - 12,5 = 50,5$ The *Difference* of the *Extreams*.

And $11 - 1 = 10$ The *Number* of *Terms* less 1.

Then 10) $50,5$ ($5,05 = 5$ l. 1 s. The *Common Difference*.

l. s. l. s. l. s.

Consequently $12 . 10 + 5 . 1 = 17 . 11$ The Second Payment.

l. s. l. s. l. s.

And $17 . 11 + 5 . 1 = 22 . 12$ The Third Payment, &c.

Example 3.

A Man is to Travel from London to a certain Place in Ten Days, and to go but Two Miles the first Day, encreasing every Days Journey by an equal Excess; so that the last Days Journey may be Twenty Nine Miles; what will each Days Journey be, and how many Miles is the Place he goes to, distant from London.

First $29 - 2 = 27$ The *Difference* of the *Extreams*.

And $10 - 1 = 9$ The *Number* of *Terms* less 1.

Then $9 \times 27 = 243$ The *Common Difference*.

Consequently $2 + 3 = 5$ The *second Days Journey*.

And $5 + 3 = 8$ The *Third Days Journey, &c.*

Again $29 + 2 = 31$ The *Sum* of the *Extreams*.

10 The *Number* of *Terms*.

$2 \times 310 = 620$ The *Distance Required*.

There are Eighteen *Theorems* more relating to Questions in *Arithmetick Progression*; but because they would Require a great many Words to shew the Reason of them: therefore refer the Reader to the Second Part, viz. The *of Algebra*; where he may find their *Analitical Investigation*.

Sect. 2. Concerning *Geometrical Proportion Continued* sometimes called *Geometrical Progression*.

When a *Rank* or *Series* of *Numbers* do either Increase by one Common *Multiplicator*, or Decrease by one Common *Divisor*; those *Numbers* are said to be in *Geometrical Proportion* continued.

As $\begin{cases} 2 . 4 . 8 . 16 . 32 . \&c. \text{ here } 2 \text{ is the common } \textit{Multiplier} \\ 64 . 32 . 16 . 8 . 4 . \&c. \text{ here } 2 \text{ is the common } \textit{Divisor} \end{cases}$

Or $\begin{cases} 2 . 6 . 18 . 54 . 162 . \&c. \text{ here } 3 \text{ is the common } \textit{Multiplier} \\ 162 . 54 . 18 . 6 . 2 . \text{ here } 3 \text{ is the common } \textit{Divisor} \end{cases}$

Note, the common *Multiplier* (or *Divisor*) is called the *Ratio*; and it shews the *Habitude* or *Relation* the *Numbers* have to one another. viz. whether they are Double, Triple, Quadruple, &c. Which *Euclid* thus defines.

Ratio (or *Rate*) is the mutual *habitude* or respect of two *Magnitudes* (consequently two *Numbers*) of the same kind each to other, according to quantity. Eu. 5. Def. 3.

Proportion (rather *Proportionality* or *Analogy*) is a *Similitude* of *Ratio's*. Eu. 5. Def. 4.

So that there cannot be less than three *Terms* to form *Proportionality* or *Similitude* of *Ratio's*; and if but three *Terms*, the second must supply the place of two. As in the $2 . 4 . 8$. That is, $2 : 4 :: 4 : 8$. (of $::$ See Page 5)

Here 4 the *Middle Term* supplies the place of two *Terms* to wit, of the second and third; 8 bearing the same *Ratio* to 4 as 4 bears to 2.

on, likeness or *Proportion* to 4. As 4 doth to 2. *Viz.* As 4 : Is to 4 : : So is 4 : To 8.

Lemma 1.

If three *Numbers* are proportional, the *Rectangle* or *Product* of the two *Extreams*; *viz.* of the first and last *Terms*, will be equal to the *Square* of the *Mean* or *Middle Term*. (20 *Euclid.* 7)

As in these $2 : 4 :: 4 : 8$ Here $8 \times 2 = 16$ the *Product* of the *Extreams*.

And $4 \times 4 = 16$ the *Square* of the *Mean*. *Ergo* $8 \times 2 = 4 \times 4$.

Coral 1.

Hence it follows, that if the *Product* of any two *Numbers*, be equal to the *Square* of a third *Number*; those three *Numbers* will be in *Proportion*.

Lemma 2.

If four *Numbers* are proportional, the *Product* of the two *Extreams*, will be equal to the *Product* of the two *Means*. (19 *Euclid.* 7)

As in these, $2 : 4 :: 8 : 16$. Here $16 \times 2 = 32$

And $8 \times 4 = 32$. Consequently $16 \times 2 = 8 \times 4$.

Coral 2.

From hence it follows, that if the *Product* of any two *Numbers*, be equal to the *Product* of any other two *Numbers*, those four *Numbers* are *Proportionals*.

And from these two *Lemma's*, it will be easie to conceive, that if never so many *Numbers* are in continued *Proportion*; the *Product* of the two *Extreams*, will be equal to the *Product* of any two *Means*, that are equally distant from the *Extreams*.

As in these $2 . 4 . 8 . 16 . 32 . 64 . \&c.$

Here $64 \times 2 = 32 \times 4 = 16 \times 8 . \&c.$ And if the *Number* of *Terms* be odd.

As in these $2 . 4 . 8 . 16 . 32 . 64 . 128 \&c.$

Then $128 \times 2 = 64 \times 4 = 32 \times 8 = 16 \times 16$.

Note. The *Character* made use on to signify continued *Proportionals* is $::$

In

In every Series of $\div$ (viz. of continued Proportionals) that Number which is compared to another, is called the *Antecedent* of the *Ratio*; and that Number to which it is compared, is called its *Consequent*.

As in these, $2 : 4 :: 4 : 8$. Here 2 is the *Antecedent*, and 4 is the *Consequent*; and 4 the middle term is an *Antecedent* to 8 its *Consequent*: Whence it follows, that in every Series of $\div$ all the Middle Terms between the first and last are both *Antecedents* and *Consequents*.

As in these, $2 . 4 . 8 . 16 . 32 . 64 . \&c.$ Here $4 . 8 . 16 . 32$ are both *Consequents* and *Antecedents*.

For $2 : 4 :: 4 : 8 :: 8 : 16 :: 16 : 32 :: 32 : 64 \&c.$

So that all the Terms except the last are *Antecedents*, And all the Terms except the first are *Consequents*.

Lemma 3.

If never so many Numbers are Proportional it will be. As any one of the *Antecedents*: Is to its *Consequent*: So will the Sum of all the *Antecedents* be; To the Sum of all the *Consequents*. (12 Euclid. 5)

That is, in the foregoing Series.

$$2 : 4 :: 2+4+8+16+32 : 4+8+16+32+64.$$

For it's evident, that $4+8+16+32+64$ the Sum of all the *Consequents*, is double to $2+4+8+16+32$ the Sum of all the *Antecedents*; As 4 is to 2. According to the *Ratio*. And would have been Triple, Or Quadruple, &c. had the *Ratio* been 3 Or 4 &c.

Note, In every Series of $\div$ the *Ratio* is found by Dividing any of the *Consequents* by its *Antecedent*.

As in these $2 : 6 :: 6 : 18 :: 18 : 54 :: 54 : 162$. Here $2 \div 6$ (3 the *Ratio*. Or $6 \div 18$ (3 &c.

From the second and third Lemma's may be raised two General Theorems or Rules, for finding the Sum of any Series in $\div$ without a continued Addition of all the Terms.

Let the Series $2 . 4 . 8 . 16 . 32 . 64 . 128 .$ be given to find its Sum.

Suppose $z =$ the Sum of all the Terms.

Then will $z - 128 =$ the Sum of all the *Antecedents*.

And $z - 2 =$ the Sum of all the *Consequents*.

But $2 : 4 :: z - 128 : z - 2$. per Lemma 3.

Ergo $4z - 512 = 2z - 4$. per Lemma 2.

Cor.

Consequently $4z - 2z = 512 - 4$.

Theorem $\left\{ z = \frac{512-4}{4-2} \right.$ In words at length thus.

Theorem 1. $\left\{ \begin{array}{l} \text{From the Product of the second and last} \\ \text{Terms, Subtract the Square of the first Term;} \\ \text{that Remainder being Divided by the second} \\ \text{Term less the first; will give the Sum of all} \\ \text{the Series.} \end{array} \right.$

Or if the first Term the common Ratio, and the last Term be only given. Then

Theorem 2. $\left\{ \begin{array}{l} \text{Multiply the last Term into the Ratio, and} \\ \text{from their Product Subtract the first Term;} \\ \text{Divide that Remainder by the Ratio less Uni-} \\ \text{ty or 1, and it will give the Sum of all the} \\ \text{Series.} \end{array} \right.$

For $4z - 2z = 512 - 4$. As above.

Consequently $2z - z = 256 - 2$ viz. the last Divided by 2.

Then $z = \frac{256-2}{2-1}$ *Theorem 2.*

Example. Let 2 . 6 . 18 . 54 . 162 . 486 . be the given Series.

Here 2 is the first Term. 3 is the Ratio, and 486 the last Term.

But $486 \times 3 = 1458$. And $1458 - 2 = 1456$.

Then $3 - 1 = 2$) 1456 (728 the Sum required.

That is $728 = 2 + 6 + 18 + 54 + 162 + 486$.

In either of these *Theorems* it is required to have the last Term known (the which in a Long Series of $\div$ will be very tedious to come at by a Continued Multiplication, &c. It will therefore be convenient to shew how to obtain either the last Term or any other Term, whose place is assigned; without producing all the Terms.

In order to that it will be necessary to premise the Coherence or Similitude that is betwixt Numbers in *Arithmetical Progression*, and those in *Geometrical Proportion*.

If to any Series of Numbers in $\div$ when the first Term is not an Unit or 1, there be Assigned a Series of Numbers in *Arithmetick Progression*, beginning with an Unit or 1, and whose Common Difference is 1. Called *Indices* or *Exponents*.

Thus, $\left\{ \begin{array}{l} 1 . 2 . 3 . 4 . 5 . 6 . 7 . \text{Indices.} \\ 2 . 4 . 8 . 16 . 32 . 64 . 128 . \text{&c. } \div \end{array} \right.$

Then

Then will the *Addition* or *Subtraction* of any two of those *Indices* (or *Numbers* in *Arithmetick Progression*) directly Correspond with the *Product* or *Quotient* of their *Respective Terms* in the *Series* of $\div$

That is, $\begin{cases} \text{As } 3+4=7 \\ \text{So } 8 \times 16=128 \end{cases}$ the *Seventh Term* in $\div$

Again, $\begin{cases} \text{As } 6+4=10. \\ \text{So } 64 \times 16=1024. \end{cases}$ The *Tenth Term* in $\div$

Or, $\begin{cases} \text{As } 7-3=4. \\ \text{So } 128 \div 8=16. \end{cases}$ Or, $\begin{cases} \text{As } 6-2=4. \\ \text{So } 64 \div 4=16 \end{cases}$ &c.

But if the *Series* of $\div$ begin with an *Unit*, the *Indices* must begin with a *Cypher*,

As in these $\begin{cases} 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \text{\&c.} \\ 1 \cdot 2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot 64. \end{cases}$

Now by help of these *Indices*, and a few of the first *Terms* in any *Series* of $\div$ It is plain that any *Term*, whose *Place* or *Distance* from the first *Term* is Assigned, may be speedily obtained, without producing the *Whole Series*.

Example 1.

A Man bought a Horse, and was to give a *Farthing* for the first Nail, two for the second, four for the third, &c. In $\div$ The *Number* of Nails was to be 7 in every Shoe, viz. 28 Nails in all. What must he have paid for the Horse.

First, $\begin{cases} 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \text{Indices.} \\ 1 \cdot 2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot \text{Farthings in } \div \end{cases}$

Then, $\begin{cases} 5+5=10 \\ 32 \times 32=1024 \end{cases}$ And $\begin{cases} 10+10=20 \\ 1024 \times 1024=1048576 \end{cases}$

Again, $\begin{cases} 4+3=7 \\ 16 \times 8=128 \end{cases}$ Lastly $\begin{cases} 20+7=27 \\ 1048576 \times 128=134217728 \end{cases}$

Which is here to be accounted the 28 and last *Term*. Because the first *Term* in the *Series* is 1. Which doth neither *Multiply* nor *Divide*.

Now this 134217728 being the *Number* of *Farthings* to be paid for the last Nail, by it the common *Ratio* which is 2, and the first *Term* which is 1, may be found the *Sum* of all the *Series*. Per *Theorem 2.*

134217728

2

268435456 From this *Product* Subtract 1.

Viz. 268435456 - 1 = 268435455. Then 2 - 1 = 1 the *Divisor*.

Consequently 268435455 is the *Sum* of all the *Series* or *Price* of the Horse in *Farthings*; which being brought into *Pounds*, &c. (See *Page* 46) will be 279620*l.* 5*s.* 3*d.* 3 *Farthings*.

Example 2.

A cunning *Servant* agreed with a *Master* (unskil'd in *Numbers*) to serve him Eleven Years, without any other *Reward* for his *Service* but the *Produce* of a *Wheat-corn* for the first Year; And that *Product* to be Sow'd the Second Year; and so on from Year to Year until the end of the Time. Allowing the *Encrease* to be but in a *Ten-fold Proportion*.

'Tis *Required* to find the *Sum* of the whole *Produce*.

First $\begin{cases} 1. 2. 3. 4. 5. \text{Indices or Years.} \\ 10. 100. 1000. 10000. 100000 \text{Wheat-corns in } \div \end{cases}$

Then $\begin{cases} \text{As } 4 + 2 = 6 \\ \text{So } 10000 \times 100 = 1000000. \text{ the 6th Years Produce} \end{cases}$

And $\begin{cases} 6 + 5 = 11 \\ 1000000 \times 100000 = 100000000000. \text{ The Eleventh or last Years Produce.} \end{cases}$

Then, either by *Theorem* 1 or 2 the *Sum* of all the *Series* will be, 11111111110 *Corns*. Now it may be *Computed* from *Page* 31 and 34, that 7680 *Wheat Corns* round and dry out of the middle of the *Ear*, will fill *Statute Pint*. If so;

Then 7680) 11111111110 (14467592 *Pints*, but 64 *Pints* are contained in a *Bushel*.

Therefore 64) 14467592 (226056 $\frac{1}{8}$ *Bushels*. Suppose it to be sold for 3 *Shillings* the *Bushel*;

Then $\begin{cases} 226056 \frac{1}{8} \\ 3 \end{cases}$

Shillings 678168 $\frac{1}{8}$ = 33908*l.* 8*s.* 4 $\frac{1}{2}$ *d.* A very good *Recompence* for Eleven Years *Service*.

There are several pretty *Questions* Resolved by *Numbers* in *Arithmetical Progression*; And by those in $\div$ which the ingenious *Learner* will easily perceive hereafter; *viz.* When we come to the *Solution* of *Questions* relating to *Interest* and *Annuities*. &c.

M

There

There is also a Third Kind of *Proportion*, called *Musical*; which being but of little or no common use, I shall therefore give but a short account of it.

Musical Proportion or *Habitude*, is, when of Three Numbers, the first hath the same *Proportion* to the Third; As the *Difference* between the First and Second hath; To the *Difference* between the Second and Third.

As in these. $6 \cdot 8 \cdot 12$ viz. $6 : 12 :: 8 - 6 : 12 - 8$

If there are Four Numbers in *Musical Proportion*, the First will have the same *Proportion* to the Fourth; As the *Difference* between the First and Second hath; To the *Difference* between the Third and Fourth.

As in these $8 \cdot 14 \cdot 21 \cdot 84$.

Here $8 : 84 :: 14 - 8 = 6 : 84 - 21 = 63$.

That is, $8 : 84 :: 6 : 63$.

The Method of finding out Numbers in *Musical Proportion*, is best expressed by Letters; as shall be shewed in the *Algebraick Part*.

Sect. 3. How to Change or Varye the Order of Things, &c.

This being a Thing not Treated of in any common Books of *Arithmetick* (that I have had the opportunity of perusing) made me think it would be acceptable to the young Learner to know how oft its possible to Varye or Change the Order or Position of any Proposed Number of Things.

As how many several Changes may be Rung upon any Proposed Number of Bells; Or how many several Variations may be made of any Determined Number of Letters; Or any other things exposed to be Varied.

The Method of finding out the Number of Changes, is by a continual Multiplication of all the Terms in a Series of Arithmetical Progressionals; whose first Term, and Common Difference is Unity or 1. And last Term the Number of Things proposed to be Varied. viz. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$ &c. As will appear from what follows.

1. If the things proposed to be varied are only two, they admit of a double Position, as to order of Place; And no more.

$$\text{Thus, } \left\{ \begin{array}{c} 1 \\ 2 \end{array} : \begin{array}{c} 2 \\ 1 \end{array} \right\} = 2 = 1 \times 2.$$

2. And if three things are proposed to be varied, they may

be changed fix several ways; As to their order of Places; And no more.

For beginning with 1 there will be $\begin{cases} 1 \cdot 2 \cdot 3 \\ 1 \cdot 3 \cdot 2 \end{cases}$

Next beginning with 2 there will be $\begin{cases} 2 \cdot 1 \cdot 3 \\ 2 \cdot 3 \cdot 1 \end{cases}$

Again, begining with 3 and it will be $\begin{cases} 3 \cdot 1 \cdot 2 \\ 3 \cdot 2 \cdot 1 \end{cases}$

Which in all make 6 or 3 times 2. viz. $1 \times 2 \times 3 = 6$.

3. Suppose four things are Proposed to be varied; Then they will admit of 24 several Changes, as to their Order of different Places.

For beginning the Order with 1 it will be $\begin{cases} 1 \cdot 2 \cdot 3 \cdot 4 \\ 1 \cdot 2 \cdot 4 \cdot 3 \\ 1 \cdot 3 \cdot 2 \cdot 4 \\ 1 \cdot 3 \cdot 4 \cdot 2 \\ 1 \cdot 4 \cdot 2 \cdot 3 \\ 1 \cdot 4 \cdot 3 \cdot 2 \end{cases}$
Here is Six Different Changes.

And for the same Reason there will be 6 Different Changes when 2 begins the Order, and as many when 3 and 4 begins the Order; which in all is $24 = 1 \times 2 \times 3 \times 4$. And by this Method of Proceeding, it may be made evident that Things admit of 120 several Variations or Changes; and 6 Things of 720. &c. As in this following Table.

The Number of Things proposed to be varied.	The manner how their several Variations are Produced.	The Different Changes or Variations every one of the Proposed Numbers can admit of.
1	1×1	$= 1$
2	1×2	$= 2$
3	2×3	$= 6$
4	6×4	$= 24$
5	24×5	$= 120$
6	120×6	$= 720$
7	720×7	$= 5040$
8	5040×8	$= 40320$
9	40320×9	$= 362880$
10	362880×10	$= 3628800$
11	3628800×11	$= 39916800$
12	39916800×12	$= 479001600$

These may be thus Continued on to any Assigned Number. Suppose to 24 the Number of Letters in the Alphabet, which will admit of 620448401733239439360000. Several Variations.

From these Computations may be started several pretty, and indeed very strange Questions.

Examples.

Six Gentlemen that were Travelling met together by Chance at a certain Inn upon the Road, where they were so pleased with their Host, and each others Company, that in a Frolick they made a Contract to stay at that Place, as long as they together with their Host, could sit every Day in a different Order or Position at Dinner; which by the foregoing Computations will be found near 14 Years. For they being made 7 with their Host, will admit of 5040 Different Positions but 5040, being Divided by $365\frac{1}{4}$ the Number of Days in one Year; will give 13 Years and 291 Days. A very pretty Frolick indeed.

I have been told (That before the great Fire which happened, Anno 1666) there was 12 Bells in St. Mary Le Bon Church in Cheap-side, London. Suppose it were Required to tell how many several Changes might have been Rung up on those 12 Bells; and at a moderate Computation how long all those Changes would have been Ringing but once over.

First, $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 = 47900160$ the Number of Changes.

Then supposing there might be Rung 10 Changes in one Minut: viz. $12 \times 10 = 120$ Stroaks in a Minut, which is 120 Stroaks in a Second of Time, now according to that rate there must be allowed 47900160 Minuts to Ring them once over in all their different Changes; viz. $479001600 \div 10 = 47900160$.

In one Year there is 365 Days, 5 Hours and 49 Minuts which being Reduced into Minuts, is 525949.

Then $525949 \div 47900160$ (91 Years, and 26 Days.

So long would those 12 Bells have been continually Ringing without any Intermiſſion, before all their different Changes could have been truly Rung but once over. 'Tis strange and seems almost incredible, that a few things should produce such Varieties.

But that which seems yet more strange and surprising, yea even impossible to those who are not a little vers'd in the power of *Numbers*; is, that if two *Bells* more had been added to the aforesaid 12 they would have advanc'd the *Number of Changes* (and consequently the time) beyond common belief. For 14 *Bells* would require (at the same rate of *Ring*ing as before) about 16575 *Years* to *Ring* all their different *Changes* but once over.

And if it were possible to *Ring* 24 *Bells* in *Changes*, and at the same Rate of 10 *Changes* in a *Minut*, which is 4 *Stroaks* in one *Second*; they would require more than 11700000000000000000000 *Years* to *Ring* them but once over in all their different *Changes*; as may easily be computed from the precedent *Table*.

CH A P. VII.

Of Proportion Disjunct commonly called the Golden Rule.

Proportion Disjunct, or the *Golden Rule*, is either *Direct*, or *Reciprocal* called *inverse*. And those are both *Simple* and *Compound*.

Sect. 1.

Direct Proportion is, when of four *Numbers*, the First beareth the same *Ratio* or *Proportion* to the Second; As the Third doth to the Fourth.

As in these $2 : 8 :: 6 : 24$.

Consequently the greater the second *Term* is, in respect to the first; the greater will the fourth *Term* be, in respect to the third.

That is, as 8 the second *Term*, is 4 times greater than 2 the first *Term*: So is 24 the fourth *Term*, 4 times greater than 6 the third *Term*.

Whence it follows, that if four *Numbers* are in *Direct Proportion*, the *Product* of the two *Extreams*, will always be equal to the *Product* of the two *Means*, as well in *Disjunct* as in continued *Proportion*; according to *Lemma 2. Page 77*.

For As $2 : 2 \times 4 :: 6 : 6 \times 4$. Or As $3 : 3 \times 5 :: 6 : 6 \times 5$.

But $2 \times 6 \times 4 = 2 \times 4 \times 6$. Or $3 \times 6 \times 5 = 3 \times 5 \times 6$.

That is, the *Product* of the *Extreams* is Equal to that of the *Means*.

Again,

Again, the less the second *Term* is, in respect to the first; the less will the fourth *Term* be, in respect to the third.

As in these $18 : 6 :: 12 : 4$.

That is, $18 : 18 \div 3 :: 12 : 12 \div 3$.

But $18 \times 12 \div 3 = 18 \div 3 \times 12$. *Viz.* $18 \times 4 = 6 \times 12$.

Consequently $2 \cdot 8 \cdot 6 \cdot 24$. And $18 \cdot 6 \cdot 12 \cdot 4$ are true *Proportionals*. per *Coral.* 2. Page 77.

From these Considerations, comes the Invention of finding a fourth *Number* in *Proportion* to any three given *Numbers*. Whence it's call'd the *Rule of Three*.

For if the second *Number* Multiplied into the third, be equal with the first Multiplied into the fourth, it is easie to conceive, that if the *Product* of the second and third be Divided by the first, the *Quotient* must needs be the fourth *Number*. For if that *Number* which Divides another, be Multiplied into the *Quotient* produced by that *Division*; their *Product* will be equal to the *Number Divided*. See Page 21.

As in these $2 : 8 :: 6 : 24$. Here $8 \times 6 = 48 = 24 \times 2$.

But if $24 \times 2 = 48$. then will $48 \div 2 = 24$. Or $48 \div 4 = 12$.

Note Any four *Numbers* in *Direct Proportion* may be varied several ways. As in these.

Viz. If $2 : 8 :: 6 : 24$. Then $2 : 6 :: 8 : 24$.

And $6 : 24 :: 2 : 8$. Or $24 : 6 :: 8 : 2$. &c.

These *Variations* being well understood, will be of no small use in the true stating of any *Question* in this *Rule of Three*.

When three *Numbers* are given, and it is required to find a fourth *Proportional*; the greatest difficulty (if there be any) will be in the right stating the *Question*, or *Abstracting* the *Numbers* out of the words in the *Question*, and placing them down in their proper order.

Now this will be very easie if it be truly considered, that always two of the three given *Terms*, are only supposed, and assign or limit the *Ratio* or *Proportion*. The third moves the *Question*; And the fourth gives the *Answer*.

As for instance; If 3 *Yards* of *Cloth* cost 9 *Shillings*: what will 6 *Yards* cost at the same *Rate* or *Proportion*.

Here 3 *Yards*, and 9 *Shillings*, are two supposed *Numbers* that imply the *Rate*; as appears by the word (if) *viz.* if 3 *Yards* cost 9 *Shillings* (Then comes the *Question*) What will 6 *Yards* cost.

N. B.

N. B. The *Term* which moves the *Question* hath generally some of these words before it; viz. *What will? How many? How long? How far? or How much? &c.*

Then (carefully observe this; viz.) The first *Term* in the *Supposition* must always be of the same kind and *Denomination* with that *Term* which moves the *Question*. And the *Term* sought will always be of the same kind and *Denomination* with the second *Term* in the *Supposition*.

Thus $\begin{matrix} \text{Yards. Shil.} & \text{Yards. Shil.} \\ 3 : 9 :: 6 : \end{matrix}$ Then

All *Questions* in *Direct Proportion* may be *Answered* by three several *Theorems*.

Theorem 1 $\begin{cases} \text{Multiply the second and third Terms together;} \\ \text{and Divide thir Product by the first Term;} \\ \text{the Quotient will be the Answer required.} \end{cases}$

Yards. Shil. Yard. Shil.

Thus $3 : 9 :: 6 : 18.$ the Answer.

$3 \overline{) 54} (18 \text{ Shillings.}$ $\begin{cases} \text{because the second} \\ \text{Term was Shillings.} \end{cases}$

Theorem 2 $\begin{cases} \text{Divide the second Term by the first, then} \\ \text{Multiply the Quotient into the third Term;} \\ \text{and their Product will be the Answer required.} \end{cases}$

Yds. Shil. Yds. Shil.

$3 : 9 :: 6 : 18.$

Thus $3 \overline{) 9} (= 3.$ Then $3 \times 6 = 18.$ as before.

Theorem 3 $\begin{cases} \text{Divide the third Term by the first, then Mul-} \\ \text{tiply that Quotient into the second Term, and} \\ \text{their Product will be the Answer.} \end{cases}$

Yds. Shil. Yds. Shil.

$3 : 9 :: 6 : 18.$

Thus $3 \overline{) 6} (= 2.$ And $9 \times 2 = 18.$ As before.

Here you see that all the three *Theorems* are equally true; but the first is most General, and usually practised. Yet the two last may be readily performed when either the second or third *Term* can be *Divided* by the first; And will be found of singular use in the *Rules of Fellowship*, &c. as will appear further on. *Quest.*

Quest. 2. If 8 lb. of Tobacco cost 14 Shillings; What will half a hundred Weight (*viz.* 56 lb.) cost at the same Rate.

Thus 8 lb : 14 s : : 56 lb : 4 l. 18 s. the Answer.

$$\begin{array}{r} 14 \\ 224 \\ \hline 56 \end{array}$$

$$8 \overline{) 784} (=98 \text{ s.} = 4 \text{ l. } 18 \text{ s.})$$

Or thus 8) 56 (=7. Then $14 \times 7 = 98 \text{ s.}$ as before.

Quest. 3. If 14 Shillings will buy 8 lb. of Tobacco; How much will 4 l. 18 s. buy after the same Rate.

Stated thus 14 s : 8 lb : : 4 l. 18 s. = 98 s : —

Then $98 \times 8 = 784$. And 14) 784 (56 lb. the Answer.

Quest. 4. If half a hundred weight of Tobacco be worth 4 l. 18 s. How much may I buy for 14 Shillings at that Rate.

Stated thus 4 l. 18 s. = 98 s : 56 lb : : 14 s : —

Then $56 \times 14 = 784$. And 98) 784 (8 lb. the Answer.

Quest. 5. Suppose 4 l. 18 s. will buy 56 lb. of Tobacco, what will 8 lb. of the same Tobacco cost.

This Question is thus Stated 56 lb : 4 l. 18 s. = 98 s : : 8 lb : —

Then $98 \times 8 = 784$. And 56) 784 (=14 s. the Answer.

Note. The three last Questions are only the second varied, being proposed purely to give an instance how any Question in this Rule of Three may be varied, according to Page 86.

Quest. 6. What will $\frac{3}{4}$ of a Yard of Velvet cost, when the Price of 21 Yards and a half is worth 22 l. 10 s. 6 d. This Question truly Stated will stand

Thus $21\frac{1}{2} \text{ Yds} : 22 \text{ l. } 10 \text{ s. } 6 \text{ d} : : \frac{3}{4} \text{ To the Answer}$

Which may be found three several ways; *viz.* by Reduction; by Vulgar Fractions; or by Decimals.

1. By Reduction. Bring the first and third Terms into one Denomination; *viz.* into Quarters, and Reduce the second Term into its least Denomination. per Sect. 4. Page 42.

Thus $21\frac{1}{2} = 86 \text{ Quarters}$. And $22 \text{ l. } 10 \text{ s. } 6 \text{ d} = 5406 \text{ Pence}$

Then $86 : 5406 : : 3 : 15 \text{ s. } 8\frac{1}{2} \text{ d.}$ For $5406 \times 3 = 16218$

And

And 86) 16218 ($=188\frac{1}{8}d$. Then $188\frac{1}{8} Pence = 15 s.$
 $d. 2\frac{1}{4}$ Farthings; the Answer.

2. The same Question stated in *Vulgar Fractions* will stand
 Thus $21\frac{1}{2} = \frac{41}{2} : 22\frac{3}{4} = \frac{91}{4} :: \frac{1}{4} :$ (See §. 3. Page 50.)

Then $\frac{91}{4} \times \frac{1}{4} = \frac{91}{16}$. And $\frac{41}{2} \div \frac{91}{16} (= \frac{328}{91})$ Page 55. 56.

These $\frac{328}{91}$ Parts of a Pound, are brought into *Shillings*
 by Multiplying the Numerator with 20 and Dividing the
 Product by its Denominator. &c.

Thus $5406 \times 20 = 108120$. And 6880) 108120 (15 s.

And there Remains 4920. Again $4920 \times 12 = 59040$.

Then 6880) 59040 (8 d. and $\frac{1}{8} c$. As before.

3. The same wrought by *Decimal Fractions*, will be thus ;

$21\frac{1}{2} = 21,5$ 22 l. 10 s. 6 d. = 22,525 and $\frac{1}{4} = 0,75$

Therefore $21,5 : 22,525 :: 0,75 :$ to the Answer.

Then $22,525 \times 0,75 = 16,89375$

And 21,5) 16,89375 (0,7857 l. = 15 s. 8 d. 2 far. $\frac{21}{1000}$

Quest. 7. If 2 C. 3 qrs. 21 lb. of Sugar, cost 6 l. 1 s. 8 d.
 what will 12 C. 2 qrs. cost at the same Rate.

That is 2 C. 3 qr. 21 lb. : 6 l. 1 s. 8 d. :: 12 C. 2 q : To what?

$\begin{array}{r} 4 \\ 11 \text{ qrs.} \\ 28 \\ \hline 88 \\ 22 \end{array}$	$\begin{array}{r} 20 \\ 121 \text{ s.} \\ 12 \\ \hline 250 \\ 121 \end{array}$	$\begin{array}{r} 4 \\ 50 \text{ qrs.} \\ 28 \\ \hline 1400 \text{ lb.} \end{array}$
--	--	--

ix. $308 + 21 = 329 \text{ lb.} : 1460 d :: 1400 \text{ lb.} :$

Then $1460 \times 1400 = 2044000$. And 329) 2044000 (6212 d.
 = 25 l. 17 s. 17 s. $8\frac{1}{4}d$. the Answer required.

The same Question Stated in *Decimals*, will stand

Thus $2,9375 : 6,0833 :: 12,5 :$ To the Answer.

Then $6,0833 \times 12,5 = 76,04125$ which being Divided
 by 2,9375 will give 25,8863 &c. the Answer in *Decimals*,
 which brought into *Coin*, will be 25 l. 17 s. $8\frac{1}{4}d$. As before.

Note, When the first Term is an Unit or 1 the Question
 Answered by Multiplication only.

Example. Suppose I give 5 Shillings 4 Pence for one Ounce
 Silver, what must I pay for $32\frac{1}{2}$ Ounces at the same
 rate.

That is $1 \text{ } \overline{3} : 5 s. 4 d :: 32\frac{1}{2} \text{ } \overline{3} :$ To &c.

Which is best Stated thus $1 : 64 d :: 32,5 :$

N

Then

Then $32,5 \times 64 = 2080 \text{ d.} = 8 \text{ l. } 13 \text{ s. } 4 \text{ d.}$ the Answer required. For 1 neither *Multiplies* nor *Divides*.

When the second or third Term is an Unit or 1 then the Question is Answered by Division only. As in this Example.

If a Silver Tankard weighing 21 Ounces, cost 5 l. 19 s. what's that an Ounce.

Thus $21 \text{ oz.} : 5 \text{ l. } 19 \text{ s.} = 119 \text{ s.} :: 1 : 5 \text{ s. } 8 \text{ d.}$ To the Answer.

That is 21) 119 ($= 5 \text{ s. } \frac{1}{2} \frac{4}{1} = 5 \text{ s. } 8 \text{ d.}$

The Proof of all Questions in the Rule of Three Direct, may be easily conceiv'd from what hath been already said; viz. That the Product of the first and fourth Terms, must always be equal to the Product of the second and third Terms.

Or otherwise, by varying the Question, as in the Second, Third, Fourth and Fifth Questions.

I shall conclude this Section, with inserting a few Questions and their Answers; leaving their Work for the Learners Practice.

Quest. 1. What will the Carriage of 17 C. 3 qrs. 11 lb. come to, at the Rate of 7 s. the Hundred.

Answer. 6 l. 4 s. $11 \frac{1}{4} \text{ d.}$

Quest. 2. If 6 l. 4 s. $11 \frac{1}{4} \text{ d.}$ be paid for the Carriage of 17 C. 3 qrs. 11 lb. what was paid for the Carriage of one lb.

Answer. 3 Farthings.

Quest. 3. A Grocer bought 3 C. 1 qr. 14 lb. weight of Cloves, at the rate of 2 s. 4 d. per lb. and sold them for 52 l. 14 s. Whether did he gain or lose by the Bargain, and how much.

Answer he gained 8 l. 12 s.

Quest. 4. A Draper bought of a Merchant eight Packs of Cloth; every Pack had four Parcels in it; And each Parcel contained ten Pieces; Every Piece was twenty six Yards; He gave after the rate of four Pounds sixteen Shillings for six Yards. What came the eight Packs to, and what was it worth per Yard.

Answer. They came to 6656 l. And is worth 16 s per Yard.

Quest. 5. A Merchant bought 436 Yards of broad Cloth for 8 s. 6 d. per Yard. And sold it again for 10 s. 4 d. per Yard. What did he gain by the 436 Yards.

Answer. he gain'd 39 l. 19 s. 4 d.

Quest.

Quest. 6. A Goldsmith bought a Wedge of Gold, which weighed 14 lb. 3 oz. 8 pwt. for 514 l. 4 s. What did he pay per Ounce.

Ans. 3 l. per Ounce.

Quest. 7. What will 48 oz. 17 pwt. 20 Grains of Silver Plate come to, at the Rate of 5 s. 6 d. per Ounce.

Ans. 13 l. 8 s. 4 $\frac{1}{4}$ d.

Quest. 8. If in four Weeks one spend 13 s. 4 d. How long will 53 l. 6 s. last at that Rate.

Ans. 6 Years 47 Days, 2 Hours, 24.

Quest. 9. What will the one eight part of a Ship be worth; when the half is valued at 1015 l. 10 s.

Ans. 253 l. 17 s. 6 d.

Quest. 10. The Sun is said to perform one entire Revolution, (or three Hundred and Sixty Degrees) in the space of three Hundred Sixty five Days, five Hours, forty eight Minuts, and fifty seven Seconds of Time, called a Tropical or Solar Year. How much doth it move in one Day.

Ans. 59 . 8 . 19 &c.

Quest. 11. If $\frac{1}{8}$ of a Yard of Velvet cost $\frac{2}{3}$ of a Pound Sterling; what will $\frac{1}{16}$ of a Yard cost of the same Velvet, at that Rate.

Ans. $\frac{1}{8} \times \frac{2}{3} = 1$ s. 4 d.

Quest. 12. Suppose 2 l. and $\frac{3}{8}$ of $\frac{1}{3}$ of a Pound Sterling; will buy 3 Yards and $\frac{2}{3}$ of $\frac{1}{3}$ of a Yard of Cloth; How much will $\frac{1}{4}$ of a Yard cost, at that Rate.

Ans. $\frac{2}{4} \times \frac{2}{3} \times \frac{1}{8}$ of a Pound = 9 s. 4 $\frac{1}{2}$ d.

Sect. 2. Of Reciprocal Proportion; usually called the Rule of Three Inverse.

Reciprocal Proportion is, when of four Numbers the third (viz. that which moves the Question) beareth the same Ratio to the first; As the second does to the fourth.

Therefore, the less the third Term is, in respect to the first; The greater will the fourth Term be, in respect to the second.

Example 1.

If sixteen Men can do a piece of Work in six days; How many Days must eight Men require to do the same Work, at the same Rate of working.

Here 'tis plain that Eight Men must needs have more time than 16 Men to do the same Work. Consequently the

greater the Third Term is, in respect to the First, the lesser will the Fourth Term be, in respect to the Second.

Example 2. If 8 Men can do a piece of Work in 12 Days; how many Days will 16 Men require to do the same Work. Here it is plain the Fourth Term must be less than the Second, because 16 Men undoubtedly can do the same Work in less Time than 8 Men can.

From these Considerations, compared with those in Page 85, 'Twill be easie to perceive whether the Terms of any proposed Question are in Direct or Reciprocal Proportion.

For when according to the true Meaning or Design of any Question in Proportion. More Requires More, or Less Requires Less, the Terms are in Direct Proportion, as in the last Section.

But if More Require Less, or Less Require More (as above) then the Terms will be in Reciprocal Proportion.

The manner of placing down the Proposed Terms is the same in both Rules. viz. The First Term in the Supposition must be of the same Kind and Denomination, with the Third Term which moves the Question; and the Term sought must be of the same Kind and Denomination, with the Second Term in the Supposition. As in the two last Examples.

		Men	Days	Men	Days
Thus, in	{	Example 1.	16 : 6 ::	8 :	
		Example 2.	8 : 12 ::	16 :	

The Question being truly Stated, observe this Theorem.

Theorem { Multiply the first and second Terms together,
and Divide their Product by the Third Term,
the Quotient will be the Answer Required.

Thus in the Second Example $12 \times 8 = 96$.

Then 16) 96 (=6 Days the Answer Required.

That is, 16 Men may do the same Work in 6 Days as 8 Men can do in 12 Days.

Now the Reason of this Operation, (and consequently of the Theorem) is grounded upon this Consideration; viz. If 8 Men Require 12 Days to do the Work, 'tis plain that one Man would Require 8 times 12 Days = 96 Days to do the same Work, but if one Man can do it in 96 Days, most certain 16 Men can do it in one 16th part of that Time. Therefore 96 Divided by 16 will give the Answer Required. viz. 16) 96 (6 As before, &c.

Quest. 3. Suppose 800 Soldiers were Besieged in a Town, and their Victuals were computed to serve them two Months (or 56 Days) How many of those Soldiers

must depart the Garison, that the same Victuals may serve the remaining Soldiers 5 Months.

The Question truly Stated, will stand

Thus $\begin{matrix} \text{Month Soldier} & \text{Month Soldier.} \\ \{ & 2 : 800 :: 5 : \end{matrix}$

5) 1600 (320 So many Soldiers may stay in the Garison.

Consequently $800 - 320 = 480$ Soldiers that must go out of the Garison, which is the *Answer Required*.

Question 4. A Borrowed of his Friend B. 250*l.* for six Months, promising to do him the like Kindness upon Demand: Some time after B desires A to lend him 400*l.* the Question is, how long B may keep the 400*l.* to be fully tisfied for his former Kindness to A.

Answer 3 Months 21 Days.

Thus, $250*l.* : 6 Months :: 400*l.* : \rule{1cm}{0.4pt}$

400) 1500 (3 Months.

12

3

28

Days in one Month.

4) 84 (21 Days.

Question 5. If a Penny White Loaf ought to weigh Eight Ounces Troy Weight, when Wheat is sold for six Shillings Six-pence the Bushel, what must it weigh when Wheat is sold for four Shillings the Bushel.

Answer 10 $\frac{3}{4}$. 18 pwt. . 8 Gr.

Thus $6*s.* 6*d.* = 78*d.* : 8*oz.* :: 4*s.* = 48*d.* To the Answ:$

48) 624 (10 $\frac{3}{4}$.

Remains

44

20

= Penny Weights in one Ounce.

48) 880 (18 Pwt.

Remains

16

24

= The Grains in 1 Pwt.

48) 384 (8 Pwt.

The Proof of this *Inverse Rule* is easily Deduced from its Operations; viz. The *Product* of the First and Second Terms

Terms, must be equal to the *Product* of the Third and Fourth *Terms*.

Note. Any Question that falls under this *Inverse Rule* or *Reciprocal Proportion*, may be so Stated as to have its *Terms* in *Direct Proportion*; by only changing the Places of the First and Third Terms in the *Question*. Thus,

Question 6. If a Field will Feed Eighteen Horses for Seven Weeks, how long will it feed Forty Two Horses at the same Rate of Feeding.

First, 18 Horses : 7 Weeks :: 42 Horses : 3 Weeks.

Here the *Terms* are Stated *Inversely*, as before.

Otherwise thus 42 Horses : 7 Weeks :: 18 Horses : 3 Weeks.
Then $18 \times 7 = 126$. And $126 \div 42 = 3$ Weeks. The Answer Required.

SECT. 3. Of Compound Proportion; commonly called the Double Rule of Three.

Compound Proportion (as 'tis here meant) is, when there are Five Numbers given to find out a Sixth Proportional; And this is generally perform'd by a *Double Position*; that is, by Stating and Working the Question at two *Operations*, either in *Direct* or *Reciprocal Proportion*, according as the Question requires.

And therefore its called the Double Golden Rule ; or Double Rule of Three.

The *Double Rule Direct* is, when the *Sixth Term*, or *Number* sought, is found by two *Operations*, both of them in *Direct Proportion*.

Example 1. If a Hundred Pounds gain Six Pounds Interest in Twelve Months; how much will Three Hundred Pounds gain in Nine Months; at the same Rate.

First 1001 : 61 :: 3001 : 181.

100) $\frac{6}{1800}$ (18% } The Interest of 3000
for Twelve Months.

Months Months
Then, 12 : 18l : : 9 : 13l. 10s.

12) $\frac{9}{162}$ (134. 10 s. The Answer Required.

I suppose the Learner will easily conceive the Reason of these two Operations. For, first it's plain by Direct Proportion, that if 100*l.* gain 6*l.* in Twelve Months. 300*l.* will gain 18*l.* in the same Time, and at the same Rate.

And

And by the same *Rule* 'tis plain, that if 12 Months will Produce or give 18*l.* Interest for 300*l.* then 9 Months must needs give 13½ for the same *Sum.* viz. 300*l.*

The *Double Rule of Three Inverse* is, when the Sixth Term or Number sought is found at two Operations (as before.) But one of them Requires an Answer in *Reciprocal Proportion*.

Question 2. If Six Bushels of Oats will serve four Horses eight Days, how many Days will twenty one Bushels serve sixteen Horses, at the same Rate of Feeding.

This Question being parted into two *Positions*, the first will be thus :

If 6 Bushels of Oats will serve 4 Horses 8 Days, how many Days will 21 Bushels serve them.

Here 'tis plain that 21 Bushels will serve them longer than 6 Bushels; therefore the first *Position* falls in *Direct Proportion*.

$$\text{Thus } \left\{ \begin{array}{l} \text{Bush. Days} \\ 6 : 8 :: 21 : 28 \end{array} \right.$$

$$6 \overline{) 168} \text{ (28 Days.}$$

That is, if 6 Bushels will serve 4 Horses 8 Days 21 Bushels will serve them 28 Days.

The next *Position* must be to find how long the said 21 Bushels will serve 16 Horses at the same Rate of Feeding: 'Tis plain that 21 Bushels cannot serve 16 Horses so many Days as they will serve 4 Horses; therefore this Second *Position* falls in *Reciprocal Proportion*.

$$\text{Thus } \left\{ \begin{array}{l} \text{Horses Days} \\ 4 : 28 :: 16 : 7 \end{array} \right. \text{ The Answer Required.}$$

After the like manner any Question in the *Double Rule of Three* may be Answered by two Single *Positions*, if Care be taken in Stating them right, viz. Whether their Operations must be performed by the Single Rule *Direct*, or *Inverse*.

But all Questions in this *Double Rule*, where Five Numbers are Proposed to find a Sixth, may more easily and readily be Answered by one *General Theorem*; which compriseth both the *Direct* and *Inverse Rules*: Without considering either of them, being Deduced from the Single Operations before-going.

But first you must carefully Note, that in all Questions of this Nature Three of the Five Proposed Terms are always

ways Conditional And Supposed; And that the other two moves the Question. As for Instance in *Example 1.*

Viz. If 100*l.* will gain 6*l.* in 12 Months: These Three Terms are only supposed or Conditional. Then comes the Question? What will 300*l.* gain in 9 Months. Now, in order to raise the *General Theorem*, let us suppose, instead of the *Numbers* these *Letters*.

Viz. Let $\begin{cases} P = 100. & \text{The Principal.} \\ T = 12. & \text{The Time.} \\ G = 6. & \text{The Gain.} \end{cases}$ } In the Supposition of any Proposed Question.

And $\begin{cases} p = 300. & \text{The Principal.} \\ t = 9. & \text{The Time.} \\ g = 13,5 & \text{The Gain.} \end{cases}$ } The Three Terms wherein the Question lies.

Then $P : G :: p : \frac{Gp}{P}$ } The Product of the two Means Divide by the first Extrem.

That is $100 : 6 :: 300 : \frac{300 \times 6}{100} = 18$. Which is the first part of the Question.

Then $T : \frac{Gp}{P} :: t : g$ } Which is the second part of the Question.
Viz. $12 : 18 :: 9 : 13,5$

Ergo $Tg = \frac{Gpt}{P}$ } That is the Product of the Extremes is equal to that of the Means.

Consequently $TgP = Gpt$. Is the *Theorem*.

This *Theorem* affords two *Rules* by which all *Questions* in this double *Rule of Three*; or rather of Five *Numbers* may be Resolved; due regard being had to the true placing down of the Proposed *Terms*, which must be thus.

Always place the Three *Conditional Terms* in this Order; Let that *Number* which is the Principal Cause of *Gain*, *Loss*, or *Action*, &c. (*viz.* *P.*) be put in the First Place; That *Number* which denotes the Space of *Time* or *Distance* of Place, &c. (*viz.* *T.*) be put in the Second Place. And that *Number* which is the *Gain*, *Loss*, or *Action*, &c. (*viz.* *G.*) be put in the Third Place. Now according to these Directions the *Conditional Terms* of the last Question will stand thus; *P . T . G.*

That done, place the other Two *Terms* which move the Question, underneath those of the same Name.

Thus $\begin{cases} P . T . G. \\ p . t . \end{cases}$

Then

Then if the *Blank* or *Term* sought, fall under the Third Place, as in this Question.

It will be $\left\{ \frac{G p t}{T p} = g. \right.$ Which gives this Rule.

Rule 1. $\left\{ \begin{array}{l} \text{Multiply the Three last Terms together for a} \\ \text{Dividend; and the Two first together for a Di-} \\ \text{visor, the Quotient arising from them will be the} \\ \text{Sixth Term.} \end{array} \right.$

That is, in our Proposed Example 1.

Thus $6 \times 300 \times 9 = 16200$ The Dividend.

And $100 \times 12 = 1200$ The Divisor.

Then $1200) 16200 (13\frac{1}{2}$ The Answer. As before.

But if the *Blank* or *Term* sought, fall under the First Place then

It will be $\left\{ \frac{T g P}{t G} = p \right.$

Or if the *Blank* fall under the Second Place.

It will be $\left\{ \frac{T g P}{G p} = t \right.$ Either of these give this Rule

Rule 2. $\left\{ \begin{array}{l} \text{Multiply the first, second and last Terms to-} \\ \text{gether for a Dividend: And the other two together} \\ \text{for a Divisor; the Quotient arising from them} \\ \text{will be the Sixth Term.} \end{array} \right.$

And because our Example 2. Falls under the Consideration both of Direct and Reciprocal Proportion, let it be here propos'd again.

Viz. If 6 Bushels of Oats will serve 4 Horses 8 Days; how many Days will 21 Bushels serve 16 Horses. &c.

If the Terms of this Question be placed down as before directed they will stand;

Thus $\left\{ \begin{array}{ccc} \text{Horses Days Bushels} \\ 4 . 8 . 6 & \text{Terms in the Supposition.} \\ 16 & 21 \end{array} \right.$

Here the Blank falls under the second Place, therefore it must be found by the second Rule.

Thus $4 \times 8 \times 21 = 672$ the Dividend.

And $16 \times 6 = 96$ the Divisor.

Then $96) 672 (7$ the Answer as before
Quest.

Quest. 3. What Principal or Stock, will Gain 20*l.* in 8 Months at 6 per Cent. per Annum.

Prin : Time Gain

100 . 12 . 6 Terms in the Supposition.
 . 8 . 20

In this Question the Blank falls under the first Place, therefore it must be found by the second Rule.

Thus $100 \times 12 \times 20 = 24000$ the Dividend.

And $8 \times 6 = 48$ the Divisor.

Then 48) 24000 (500 *l.* the Answer required.

The Proof of all Questions in this double Rule of five Numbers, is best perform'd by varying the Question; viz. by stating it in another Order, as in the last Example Thus.

If 100 *l.* Gain 6*l.* in 12 Months, what will 500 *l.* Gain in 8 Months.

The Answer to this Question must be 20 *l.* if the Work of the last Example be true.

Prin. Time Gain

Stated thus $\left\{ \begin{array}{l} 100 . 12 . 6 \\ 500 . 8 . \end{array} \right\}$ then per Rule 1.

$500 \times 8 \times 6 = 24000$. And $100 \times 12 = 1200$

Then 1200) 24000 (20. the Answer &c.

Quest. 4. If 2 Men can do 12 Rods of Ditching in 6 Days, How many Rods may be done by 8 Men in 24 Days, at the same Rate of working.

Ans. 192 Rods.

Quest. 5. If the Carriage of 5 C. 3 qr. weight, 150 Miles, cost 3*l.* 7*s.* 4*d.* What must be paid for the Carriage of 7 C. 2 qr. 25 lb. weight, 64 Miles; at the same Rate.

Ans. 1*l.* 18*s.* 7½*d.* &c.

Quest. 6. If 8 Men deserve 2*l.* wages for 5 days work, How much will 32 Men deserve for 24 days; at the same Rate.

Ans. 38*l.* 8*s.*

Quest. 7. Suppose a Hundred Pounds would defray the Expences of five Men for twenty two Weeks and six Days How long would twelve Men be in spending of one Hundred and fifty Pounds, and at the same Rate.

Ans. 14 Weeks and 2 Days.

C H A P

C H A P. VIII.

Of Trading in Company; usually called the Rule of Fellowship; Also Wattering, and Exchanging of Coins &c.

The Rule of Fellowship, is that by which the Accompts of several Partners Trading in a Company, are so adjusted or made up, that every Partner may have his just part of the Gain, or sustain his just part of the Loss; according to his proportion or share of Money he hath in the Joynt-Stock: Now this falls under two Considerations, called the single and double Rules of Fellowship.

Sect. 1. The Single Rule of Fellowship; viz. That without Time.

By the Single Rule of Fellowship is adjusted the Accompts of those Partners that put all their several and perhaps different Sums of Money, into a common Stock at one and the same time; and therefore it's usually called the Rule of Fellowship without Time: Now all Questions of this Nature are Answered by so many several Operations in the Rule of Three direct, as there are Partners in the Stock.

For, As the Total Sum of Money in the Stock; Is in Proportion to the whole Gain, or Loss; So is every Man's particular part of that Stock; To his particular share of that Gain, or Loss.

Quest. 1. There are three Partners, Suppose A, B, and C, make a Joint-Stock of 96*l.* in this manner.

A, puts in 24*l.* B, puts in 32*l.* and C, puts in 40*l.* with this 96*l.* they Trade and Gain 12*l.* 'Tis required to find each Man's true Part of that Gain.

The first Operation for A's Part of the Gain will stand,

Thus $96l : 12l :: 24l : 3l = A's \text{ Gain.}$

Secondly $96l : 12l :: 32l : 4l = B's \text{ Part of the Gain.}$

Again $96l : 12l :: 40l : 5l = C's \text{ Part of the Gain.}$

Proof $3l + 4l + 5l = 12l$ the whole Gain.

That is, if the Sum of each Man's particular Gain, amount to the whole Gain, the Work is true; if not, some Error is committed which must be found out.

Note. These Operations will be very much abbreviated, if you work them by Theorem 2. Page 87. For here 96, is a common Antecedent, and 12 is the common Consequent in all the three Proportions.

Therefore $96:12::1:0,125$ is a common *Multiplier*.

Then $24 \times 0,125 = 3l.$ for *A's* Part

And $32 \times 0,125 = 4l.$ for *B's* Part } As before.

Again $40 \times 0,125 = 5l.$ for *C's* Part

Now this Method is more readily perform'd than the other, especially when the Partners are many; because one *Single Divison* serves for all the Work.

Quest. 2. Three *Merchants*, *A*, *B*, and *C*, Fraught a Ship with 248 *Tun* of Wine: thus, *A* Loaded 98 *Tun*, *B* 86 *Tun*, and *C* 64 *Tun*. By Extreimity of Weather the Seamen were forc'd to cast or throw 93 *Tun* of it over board. How much of this Loss must each *Merchant* sustain.

First $248 : 93 :: 1 : 0,375$ the common *Multiplier*,

Then $98 \times 0,375 = 36,75$ for *A's* Loss.

And $86 \times 0,375 = 32,25 = B's$ Loss.

Again $64 \times 0,375 = 24, = C's$ Loss.

Proof. $93,00 =$ the whole Loss.

Now if the *Question* were to find how much of the Remaining Wine that was saved, belongs to *A*, to *B*, and to *C*.

Then $98 - 36,75 = 61,25$ belongs to *A*.

And $86 - 32,25 = 53,75$ belongs to *B*.

And $64 - 24, = 40$ belongs to *C*.

That is, *A* ought to have 61 *Tun* and 63 *Gallons*. *B* ought to have 53 *Tun* and 189 *Gallons*. And *C* ought to have 40 *Tun* of what was Left.

Quest. 3. Suppose six Men, viz. *A*, *B*, *C*, *D*, *E*, and *F* make a Joint-Stock of 2558*l*.

	<i>l.</i>	<i>s.</i>	<i>Decimals.</i>
Thus { <i>A</i> puts in	654	. 10	= 654,5
<i>B</i> ———	543	. 15	= 543,75
<i>C</i> ———	480	. 00	= 480,
<i>D</i> ———	254	. 10	= 254,5
<i>E</i> ———	365	. 05	= 365,25
<i>F</i> ———	260	. 00	= 260,

The whole Stock 2558 . 00 = 2558,00 according to the *Question*.

With

With this Stock of 2558*l.* they Trade Eighteen *Months*, and Gain 831*l.* 7*s.* 'Tis required to find every Man's Part or Share of that Gain.

Note, *Altho' the time of Trading, viz. Eighteen Months be mentioned in the Question; yet it's no way concern'd in answering of it; as you may observe by the following Work.*

First. 2558*l.* : 831,35*l.* :: 1*l.* : 0,325 *Decimal Parts*,
Consequently, 1*l.* : 0,325 :: 654,5 : 212,7125. That is,

A's Stock 654,5 × 0,325 = 212,7125 for A.

B's Stock 543,75 × 0,325 = 176,71875 for B.

C's Stock 480, × 0,325 = 156,000 for C.

D's Stock 254,5 × 0,325 = 82,7125 for D.

E's Stock 365,25 × 0,325 = 118,70625 for E.

F's Stock 260, × 0,325 = 84,5 for F.

	<i>l.</i>	<i>parts</i>	<i>l.</i>	<i>s.</i>	<i>d.</i>
That is {	A Gains	212,7125	=	212	. 14 . 03
	B ———	176,71875	=	176	. 14 . 04½
	C ———	156,0000	=	156	. 00 . 00
	D ———	82,7125	=	82	. 14 . 03
	E ———	118,70625	=	118	. 14 . 01½
	F ———	84,5	=	84	. 10 . 00

Proof, Sum 831,35 = 831 . 7 . 00 the Gain

I have omitted resolving this Question according to the usual Method, (as before directed) of finding every Man's particular part of the Gain by the *Golden Rule*, as in the first Work of Example 1. Leaving that for the Learners Practice.

Secl. 2. The Double Rule of Fellowship; or that with Time.

This is usually called the *Double Rule of Fellowship*, because every particular Man's Money is to be considered with relation to the time of its Continuance in the Joint-Stock.

Question 1. A, and B, join in Partnership upon these Terms, *viz.* A. agrees to lay down 100*l.* and to imploy it in Trade 3 *Months*: Then B, is to lay down his 100*l.* and with the whole Stock of 200*l.* they are to Trade 3 *Months* more. Now at the end of that time, they find their whole Gain to be 21*l.* 'Tis required to know what each Man's part of the Gain ought to be, according to his Stock, and the time of imploying it.

Here

Here it is but reasonable to conclude, that *A.* ought to Gain more than *B.* notwithstanding their Stocks of Money are equal; because *A.* imploy'd his Money a longer Time than *B.*

Now for Solving of this Question, Let us suppose *A.*'s 100*l.* imployed the first 3 Months to Gain $Z =$ a Sum as yet unknown; then it must Gain $2Z$, in 6 Months; and to find what *B.* must Gain it will be.

1. Months.

$$\begin{array}{lcl} 100 & : & 6 & : & 2Z = A's \text{ Gain} \\ 100 & : & 3 & : & \text{To } B's \text{ Gain} \end{array} \left. \vphantom{\begin{array}{lcl} 100 & : & 6 \\ 100 & : & 3 \end{array}} \right\} \text{per Rule 1 Page 97.}$$

$$\text{Ergo } \frac{100 \times 3 \times 2Z}{100 \times 6} = B's \text{ Gain}$$

But *A.*'s Gain added to *B.*'s Gain must = 21*l.* the whole Gain by the Question.

$$\text{Therefore } 2Z + \frac{100 \times 3 \times 2Z}{100 \times 6} = 21 \text{ l.}$$

$$\text{That is } 100 \times 6 \times 2Z + 100 \times 3 \times 2Z = 21 \times 100 \times 6.$$

$$\text{Which contracted is } 900 \times 2Z = 21 \times 600.$$

Consequently $2Z = \frac{21 \times 600}{900}$ which gives this following Analogy.

$$\text{Viz. } 900 : 21 :: 600 : 2Z = 14 \text{ l. for } A's \text{ Gain.}$$

$$\text{And } 900 : 21 :: 100 \times 3 = 300 : 7 \text{ l. for } B's \text{ Gain.}$$

Now this way of Arguing hath not only resolved the present Question; but it also affords (and Demonstrates) a General Rule for resolving all Questions of this Nature, be the Partners never so many.

Rule { Multiply every particular Man's Stock, with the Time it is employed; then it will be, As the Sum of all those Products; Is to the whole Gain (or Loss). So is every one of those Products; To its proportional part of that whole Gain (or Loss).

Question 2. Three Merchants *A.* *B.* and *C.* enter into Partnership thus, *A.* puts into the Stock 65*l.* for 8 Months. *B.* puts in 78*l.* for 12 Months. and *C.* puts in 84*l.* for 6 Months. With these they Traffick and Gain 166*l.* 12*s.* 'Tis required to find each Man's Share of the Gain, proportionable to his Stock and time of employing it.

1. *A.*'s

1. A's Stock 65 l. * 8 Mon. the time it was employed = 520
2. B's Stock 78 l. * 12 Mon. the time it was employed = 936
3. C's Stock 84 l. * 6 Mon. the time it was employed = 504

The Sum of those Products is, 1960

Then, according to the Rule, the several Proportions will stand thus,

$$1960 : 166,6 :: 520 : 44,2 = 44 l. 4 s. 0 d. \text{ for } A.$$

$$1960 : 166,6 :: 936 : 79,56 = 79 l. 11 s. 2 \frac{1}{2} d. \text{ for } B.$$

$$1960 : 166,6 :: 504 : 42,84 = 42 l. 16 s. 9 \frac{1}{2} d. \text{ for } C.$$

The whole Gain = 166 l. 12 s. 0 d.

Or you may Work as in some of the former Examples, viz. by finding the proportional part of the Gain due to one Pound. &c.

Thus $1960 : 166,6 :: 1 : 0,085$ the common Multiplier.

$$\begin{array}{l} \text{Then } 520 \times 0,085 = 44,2 \text{ for } A. \\ \text{And } 936 \times 0,085 = 79,56 \text{ for } B. \\ \text{Also } 504 \times 0,085 = 42,84 \text{ for } C. \end{array} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{ \&c. As before.}$$

Question 3. Six Merchants, viz. A. B. C. D. E. and F. enter into Partnership, and compose a Joint-Stock in this manner;

	l.	s.		
Thus {	A puts in	64 . 10	} for {	$\left. \begin{array}{l} 4 \frac{1}{2} \\ 6 \\ 8 \frac{1}{4} \\ 12 \\ 9 \frac{1}{2} \\ 7 \end{array} \right\} \text{ Months.}$
	B	78 . 15		
	C	100 . 00		
	D	80 . 10		
	E	74 . 12		
	F	125 . 15		

They Traffick and Gain 258 l. 18 s. $4 \frac{1}{2} d.$ 'Tis required to find every Man's Share of the Gain, according to his Stock and Time it was employed.

The several Stocks of Money, and their respective times being first brought into Decimals, and then Multiplied together will produce these following Products.

	l.	Months.	
A's Stock	64,5	$\times 4,5$	the time it was employed = 290,25
B's Stock	78,75	$\times 6,$	the time it was employed = 472,5
C's Stock	100,	$\times 8,25$	the time it was employed = 825,0
D's Stock	80,5	$\times 12,$	the time it was employed = 966,0
E's Stock	74,6	$\times 9,5$	the time it was employed = 708,7
F's Stock	125,75	$\times 7,$	the time it was employed = 880,25

The Sum of those Products = 4142,7

Then

Then if you Work by the common way; it will be
 $4142,7 : 258,91875 :: 290,25 : 18,140625 = 18\text{ l. } 2\text{ s. } 9\frac{1}{4}\text{ d.}$
 for *A*'s part of the Gain; And so on for the rest.

But if you Work by the easiest way, viz. by finding the
 proportional part of the Gain due to one Pound,

Thus $4142,7 : 258,91875 :: 1 : 0,0625$

		<i>l.</i>	<i>s.</i>	<i>d.</i>	
Then	$290,25 \times 0,0625 = 18,140625$	$= 18$	$. 2$	$. 9\frac{1}{4}$	for <i>A</i>
And	$472,5 \times 0,0625 = 29,53125$	$= 29$	$. 10$	$. 7\frac{1}{2}$	for <i>B</i>
	$825, \times 0,0625 = 51,5625$	$= 51$	$. 11$	$. 3$	for <i>C</i>
	$966, \times 0,0625 = 60,375$	$= 60$	$. 7$	$. 6$	for <i>D</i>
	$708,7 \times 0,0625 = 44,29375$	$= 44$	$. 5$	$. 10\frac{1}{2}$	for <i>E</i>
	$880,25 \times 0,0625 = 55,015625$	$= 55$	$. 0$	$. 3\frac{1}{4}$	for <i>F</i>
	The whole Gain	$= 258$	$. 18$	$. 4\frac{1}{2}$	

These few *Examples* being well understood, are (I pre-
 sume) sufficient to shew the whole Business of Fellowship, &c.

Sect. 3. Of Bartering.

When *Merchants*, or *Tradesmen* exchange one Commo-
 dity for another, its called *Bartering*; and the only difficul-
 ty in this way of Dealing, lies in the due proportioning the
 Commodities to be exchanged, So, as that neither Party
 sustain Loss.

Question 1. Two *Merchants A.* and *B.* Barter; *A.* would
 exchange 5 *C.* 3 *qrs.* 14 *lb.* of *Pepper*, which is worth 3 *l.* 10 *s.*
per lb. with *B.* for *Cotton*, worth 10 *d.* *per lb.* weight. How
 much *Cotton* must *B.* give to *A.* for his *Pepper*.

Note, In order to the resolving of this Question, (and all
 other Questions of this Nature) you must first find, by the
 Rule of Three (or otherwise) the true value of that Commodity
 whose quantity is given; (which in this Question is *Pepper*.)
 And then find how much of the other Commodity will amount to
 that Sum, at the Rate proposed.

First 5 *C.* 3 *qrs.* 14 *lb.* $= 5,875$ } in Decimals.
 And 3 *l.* 10 *s.* 0 *d.* $= 3,5$ }

Then $1 : 3,5 :: 5,875 : 20,5625 = 20\text{ l. } 11\text{ s. } 3\text{ d.}$ the
 true value of the *Pepper*,

Next, Its easie to conceive, that *A.* ought to have as much
Cotton at 10 *d.* *per lb.* as will amount to 20 *l.* 11 *s.* 3 *d.* which
 may be thus found,

$10\text{ d.} : 1\text{ lb.} :: 20\text{ l. } 11\text{ s. } 3\text{ d.} = 4235\text{ d.} : 493,5\text{ lb.}$

That

That is, 4 C. 1 qr. 17½ lb. of Cotton. And so much B. must give to A. in exchange for his 5 C. 3 qrs. 14 lb. of Pepper.

Question 2. Two Merchants A. and B. Barter thus, A. hath 86 Yards of Broad Cloth worth 9 s. 2 d. per Yard; ready Money, but in Barter he will have 11 s. per Yard B. hath Shalloon worth 2 s. 1 d. per Yard ready Money 'Tis required to find how many Yards of the Shalloon B. must give to A. for his Cloth, to make his Gain in the Barter equal to that of A's.

The Method of resolving this, and the like Questions differs a little from the last Case; for in this you must first find what Advance B. ought to make per Yard upon his Shalloon, in proportion to what A. hath done upon a Yard of his Cloth.

Thus $\begin{matrix} s. & d. & d. & s. & d. & s. & d. & d. & s. & d. & d. \\ \{ & 9. & 2 = & 110 : & 11 = & 132 :: & 2. & 1 = & 25 : & 2.6 = & 30 \end{matrix}$
the advanced Price for a Yard of B's Shalloon. Then proceed as before in the last Example.

Thus 1 Yard : 11 s. :: 86 Yards : 946 s = 47 l. 6 s. the advanced value of all the Cloth.

Next, If 2 s. 6 d. will buy one Yard of Shalloon at its advanced Price, how many Yards will 47 l. 6 s. buy.

Thus 2 s. 6 d. : 1 :: 946 : 378,4 Yards.

That is, B. must give 378½ Yards of his Shalloon to A. for his 86 Yards of Broad Cloth.

These two Examples are sufficient to shew the Learner, that the Method of Bartering, or exchanging Commodities for Commodities, wholly depends upon a clear understanding of the Golden Rule, which indeed is so called, because of its Universal Use.

Sect. 4. Of Exchanging Coins.

Exchanging the Coins of one Country for those of another is like the business of Bartering Commodities. That is, it consists in finding what Sum of one Country Coin, will be equal in value to any proposed Sum of another Country Coin. And in order to perform that, it will be very Necessary to have a true Account at all times of the just values of those Foreign Coins which are to be Exchanged, as they are compared in value with our English Coin.

I say, at all times, because the Par of Exchange (as the Merchants call it) differs almost every Day from London to other Countries. That is, it rises, and falls according as Money is Plenty or Scarce; or according to the time allowed for payment of the Money in Exchange. &c.

P

Those

Those that desire to be fully satisfied in the Common values of *Foreign Coins, Weights, Measures, &c.* may find them in a Book called the *Merchants Map of Commerce*, which for brevities sake I have omitted Transcribing, and only collected these few of *Coin*.

Foreign Coins		English Coin.
		<i>l . s . d</i>
French Coin.	A Denier =	0 : 0 . 0 $\frac{1}{40}$
	12 Deniers = 1 Soultz =	0 . 0 . 0 $\frac{9}{10}$
	20 Soultz = 1 Liever =	0 . 1 . 6
	3 Lievers = 1 Crown =	0 . 4 . 6
Low Country Coin.	A Stiver =	0 . 0 . $1\frac{1}{2}$
	6 Stivers = 1 Flemish Shilling =	0 . 0 . $7\frac{1}{2}$
	20 Stivers = 1 Gilder =	0 . 2 . 0
	10 Gilders = $33\frac{1}{2}$ Shillings, } or a Flemish Pound }	1 . 0 . 0
	{ Embden Doller =	0 . 2 . $3\frac{1}{2}$
	{ Campen Doller =	0 . 2 . $7\frac{1}{2}$
	A { Zealand Doller =	0 . 3 . 0
	{ Lyons Doller =	0 . 4 . 0
	{ Specie Doller =	0 . 5 . 0
	{ Duccatoon =	0 . 6 . $3\frac{1}{2}$
Germany	{ A Rix Doller of the Empire =	0 . 4 . $5\frac{1}{4}$
	{ A Gilder of Neremberg =	0 . 7 . 1
	{ The Liever at Leghorn =	0 . 0 . 9
	{ Florence Crown Courrant =	0 . 5 . 3
In Italy and Spain.	{ Venice Ducat de Banco =	0 . 4 . 4
	{ The Currant Ducat =	0 . 3 . 4
	{ The Naple Ducat =	0 . 5 . 0
	{ The Cadiz Ducat =	0 . 5 . $6\frac{1}{4}$
	{ The Barcelona Ducat =	0 . 6 . 0
	{ The Valentia Ducat =	0 . 5 . 3
	{ The Bergonia Ducat =	0 . 4 . 4
	{ The Portugal Testoon =	0 . 1 . 3
	{ The Piece of Eight =	0 . 4 . 6

Note, The English generally reckon their Exchange with other Countries by Pence, viz. other Countries value their Crowns, Dollers, or Ducats, &c. by English Pence. Except with some parts of the Low Countries, with whom the Exchange in Pounds Sterling.

Quest. 1. How many Dollers at 4 s. 6 d. per Doller, may one have for 162 l. 18 s.

Answer 724 Dollers.

Thus

Thus $162\text{ l. } 18\text{ s.} = 3258\text{ s.}$ and $4\text{ s. } 6\text{ d.} = 45\text{ s.}$

Then $45 : 1 :: 3258 : 724$ the Answer.

Quest. 2. How many Saragosa Ducats of $5\text{ s. } 6\text{ d.}$ the Ducat may be had for 275 Bergonia Ducats, at $4\text{ s. } 4\text{ d.}$ the Piece.

Answer 216 and $3\text{ s. } 8\text{ d.}$ over

Thus $5\text{ s. } 6\text{ d.} = 66\text{ d.}$ and $4\text{ s. } 4\text{ d.} = 52\text{ d.}$

Then $275 \times 52 = 14300\text{ d.} = 275\text{ Ducats.}$

Consequently $66 \mid 14300$ ($216\frac{2}{3}$ the Answer requir'd.

Quest. 3. A Traveller would change $233\text{ l. } 16\text{ s. } 8\text{ d.}$ Sterling Money, for Venice Ducats at $4\text{ s. } 9\frac{1}{2}\text{ d.}$ per Ducat. How many Ducats must he have.

Answer 976 Ducats.

Thus $4\text{ s. } 9\frac{1}{2}\text{ d.} = 57\frac{1}{2}\text{ d.}$ and $233\text{ l. } 16\text{ s. } 8\text{ d.} = 56120\text{ d.}$

Then $57\frac{1}{2}\text{ d.} \mid 56120\text{ d.}$ (976 the Answer required.

Quest. 4. A Cassier hath Received 759 Ducats, at $7\text{ s. } 6\text{ d.}$ per Ducat; And 579 Dollers at $4\text{ s. } 8\text{ d.}$ per Doller: Which he would Exchange for Flemish Markes at $14\text{ s. } 3\text{ d.}$ per Piece, How many ought he have.

Ans. 589 Marks, and 15 d. over.

For $7\text{ s. } 6\text{ d.} = 90\text{ d.}$ and $4\text{ s. } 8\text{ d.} = 56\text{ d.}$

Then $\begin{cases} 759 \times 90 = 68310\text{ d.} & \text{the value of the Ducats.} \\ 579 \times 56 = 32424\text{ d.} & \text{the value of the Dollers.} \end{cases}$

their Sum. $= 100734\text{ d.}$

And $14\text{ s. } 3\text{ d.} = 171\text{ d.}$ the Flemish Mark in Pence,

Consequently $171 \mid 100734$ (589 &c. the Answer required,

Quest. 5. A Bill of Exchange was excepted at London for the payment of 400 l. Sterling, for the like value delivered in Amsterdam, at $1\text{ l. } 13\text{ s. } 6\text{ d.}$ for 1 l. Sterling. How much Money was delivered at Amsterdam,

Ans. 670 l. Flemish.

For $1\text{ l.} = 240\text{ d.}$ and $1\text{ l. } 13\text{ s. } 6\text{ d.} = 402\text{ d.}$

Then $240 : 402 :: 400 : 670$ The Answer required.

Quest. 6. When the Exchange from Antwerp to London is at $1\text{ l. } 4\text{ s. } 7\text{ d.}$ Flemish, for 1 l. Sterling: How many Pounds Sterling must be paid at London; to ballance 236 l. Flemish at Antwerp.

Ans. 192 l. Sterling.

Thus $1\text{ l. } 7\text{ d.} = 295\text{ d.}$ and $1\text{ l.} = 240\text{ d.}$

Then $295 : 240 :: 236 : 192$ The Answer.

Quest. 7. A Merchant delivered at London 120 l. Sterling to Receive 147 l. Flemish in Amsterdam. How much was 1 l. Sterling valued at, in Flemish Money.

Ans. 1 l. 4 s. 6 d.

Thus $120 : 147 :: 240 d : 294 d = 1 l. 4 s. 6 d. \&c.$

Quest. 8. A Factor hath sold Goods at Cadiz for 1468 pieces of Eight; valued at 4 s. 6½ d. Sterling per piece. How much Sterling Money does those pieces of Eight amount to.

Ans. 333 l. 7 s. 2 d.

Thus, if $1 = 54\frac{1}{2} d.$ then $1468 \times 54\frac{1}{2} = 80006 d. \&c.$

Quest. 9. A Traveller would have an equal Number of Crowns at 5 s. 6 d. per Crown; And Dollers at 4 s. 5 d. per Piece; How many of each sort may he have for 309 l. 8 s.

Ans. (624 of each.

Thus $309 l. 8 s. = 74256 d.$

And $5 s. 6 d. + 4 s. 5 d. = 119 d.$

Then $119 \overline{) 74256}$ (624 the Answer required.

Quest. 10. Suppose I would Exchange 527 l. 17 s. 6 d. for Dollers at 4 s. 6 d. a piece. Ducats at 5 s. 8 d. a piece, and Crowns at 6 s. 1 d. a piece. And would have 2 Dollers for 1 Ducat, and 3 Dollers for 2 Crowns. How many of each sort must I have.

Ans. 927 Dollers. 463½ Ducats, and 618 Crowns.

For. $\left\{ \begin{array}{l} 54 d. = 1 \text{ Doller.} \\ 68 d. = 1 \text{ Ducat.} \\ 73 d. = 1 \text{ Crown.} \end{array} \right\}$ per Question.

And $126690 d. = 527 l. 17 s. 6 d.$

Now if the Crowns, Dollers, and Ducats, were to be equal in Number; Then $73 + 54 + 68$ must have been the Divisor, by which 126690 must have been Divided, and the Quotient would have been the Answer to the Question. As in the last Example.

But here instead of their Sum, such parts of them must be taken as are assigned or limited by the Question; that is, the Number of some one of them may be found.

And because there must be $\left\{ \begin{array}{l} 2 \text{ Dollers for 1 Ducat.} \\ 3 \text{ Dollers for 2 Crowns.} \end{array} \right.$

Therefore, it will be but $\frac{1}{2}$ of a Ducat for one Doller, and $\frac{2}{3}$ of a Crown for one Doller.

Con:

Consequently $54 + \frac{2}{3} : + \frac{2}{3}$ of $73 = 136\frac{2}{3}$ Or $\frac{410}{3}$ will be the *Divisor* to find the *Number of Dollers*.

Thus $\frac{410}{3}$) 126690 (927 the *Number of Dollers*.

Then $\frac{1}{2}$ of $927 = 463\frac{1}{2}$ is the *Number of Ducats*.

And $\frac{2}{3}$ of $927 = 618$ is the *Number of Crowns*.

Or if you please you may form *Divisors* to find either the *Ducats* or *Crowns* first, For if it be 2 *Dollers* for one *Ducat*, and 3 *Dollers* for 2 *Crowns*, as before.

Then will 6 *Dollers* be for 3 *Ducats*, and 6 *Dollers* for 4 *Crowns*.

Therefore $\left\{ \begin{array}{l} \frac{1}{2} \text{ of a Daller} \\ \frac{3}{4} \text{ of a Ducat} \end{array} \right\}$ will be for 1 *Crown*.

Consequently $\frac{1}{2}$ of $54 : + \frac{3}{4}$ of $68 : + 73 = 205$ will be the *Divisor* to find the *Crowns* first, &c.

Quest. II. A *Cassier* is to Receive 500 l. He is offered *Crowns* at $6\text{ s. } 1\frac{1}{2}\text{ d. per Crown}$, which are worth but 6 s. Or he may have *Dollers* at $4\text{ s. } 5\text{ d. the piece}$, which are worth but $4\text{ s. } 4\text{ d.}$ Which of these shall he receive to have the least *Loss*. And how much will he lose in the *Payment*.

1 $\left\{ \begin{array}{l} 1 \text{ Crown} = 72\text{ d.} \\ 1 \text{ Daller} = 52\text{ d.} \end{array} \right\}$ according to the true Values.

2 $\left\{ \begin{array}{l} 1 \text{ Crown} = 73\frac{1}{2}\text{ d.} \\ 1 \text{ Daller} = 53\text{ d.} \end{array} \right\}$ the advanced Values.

Now to find which will be the least *Loss*; find what the advanced Value of a *Daller* ought to be in proportion, to that of 1 *Crown*.

Thus $72 : 73\frac{1}{2} :: 52 : 53\frac{22}{3}$ &c. But he may have *Dollers* at 53 d. per Piece , therefore the *Payment* in *Dollers* will be the least *Loss*, viz. 53 is less than $53\frac{22}{3}$ &c.

Next, to find what the whole *loss* will be, Divide $120000\text{ d.} = 500\text{ l.}$ by 52 and by 53 the difference of their *Quotients* will be the *Loss*.

Thus $52) 120000$ ($2307\frac{2}{3}$ And $53) 120000$ ($2264\frac{2}{3}$

Then $2307\frac{2}{3} - 2264\frac{2}{3} = 43\frac{2}{3}$ *Dollers* at $4\text{ s. } 4\text{ d.}$ are the *Loss*. viz. $9\text{ l. } 8\text{ s. } 10\frac{2}{3}\text{ d.}$

There are other ways of Answering the last *Question*, but this I take to be the easiest.

Quest. 12. Suppose I Exchange $4\text{ l. } 10\text{ s. } 10\text{ d.}$ for 11 *Crowns*, and 7 *Dollers*; And at another time I have 4 *Crowns*

Crowns and 3 *Dollers* for 1 *l.* 15 *s.* each being of the same Value with the first. What is the Value of a *Crown*, and of a *Doller*.

First 11 *Crowns* + 7 *Dollers* = 1090 *d.* } by the *Question*.
 Second 4 *Crowns* + 3 *Dollers* = 420 *d.* }

Then in order to find the Value of 1 *Crown*, you must cast off the *Dollers* by making them of the same Number Thus.

33 *Crowns* + 21 *Dollers* = 3270 *d.* the first *Mul.* with 3;

28 *Crowns* + 21 *Dollers* = 2940 *d.* the second *Mul.* with 7.

Then 5 *Crowns* = 330 *d.* being their Difference.

Consequently 5) 330 (66 = 5 *s.* 6 *d.* is the Value of 1 *Crown*.

And 4 *Crowns* = 264 *d.*

Then will 3 *Dollers* = 420 *d.* — 264 *d.* = 156 *d.*

Consequently 3) 156 (52 *d.* = 4 *s.* 4 *d.* the Value of 1 *Doller*.

CHAP. IX.

Of Alligation.

When it is required to mix several sorts of Ingredients together; As different sorts of *Corn*, *Wines*, *Wool*; *Spices*, or *Metals*; or to compose *Medicines*. &c. the Method of proportioning such Mixtures, is called the *Rule of Alligation*. And is *Divided* into two Parts or Branches; called *Medial* and *Alternate*.

SECT. I. Of Alligation Medial.

Alligation Medial, is that by which the Mean Rate or Price of any Mixture is found, when the particular Quantities of the Mixture and their Rates are given; And is thus perform'd.

First find the *Sum* of all the Quantities proposed to be mix'd; And also the *Sum* of all their particular Rates.

Then the Proportion will be.

Rule. $\left\{ \begin{array}{l} \text{As the Sum of all the Quantities : Is to the} \\ \text{Sum of all their Rates : : So is any part of} \\ \text{the Mixture : To the Mean Rate or Price of} \\ \text{that part.} \end{array} \right.$

Quest. 1. Suppose 15 *Bushels* of *Wheat* at 5 *s.* the *Bushel* and 12 *Bushels* of *Rye* at 3 *s.* 6 *d.* the *Bushel*, were mix'd

together; What is the Mean Rate or Price, it may be sold for a *Bushel* without Loss or Gain.

This *Question* prepared as Directed above will stand,

Thus $\left\{ \begin{array}{l} 15 \text{ Bushel of Wheat at } 5 \text{ s. per Bushel, comes to } 900 \text{ d.} \\ 12 \text{ Bushel of Rye at } 3 \text{ s. } 6 \text{ d. each, comes to } 504 \text{ d.} \end{array} \right.$
 $27 = \text{their Sum. And their Total Value} = 1404 \text{ d.}$

Then $27 \text{ Bush} : 1404 \text{ d} :: 1 \text{ Bush} : 52 \text{ d} = 4 \text{ s. } 4 \text{ d.}$ the Answer required.

Quest. 2. A Grocer Mixeth 36 lb. of Tobacco, worth 1 s. 6 d. a Pound, with 12 lb. of another sort at 2 s. a lb. And 12 lb. of a third sort at 1 s. 10 d. the Pound. How many he sell the Mixture per Pound.

	lb.	s.	d.	
First	$\left\{ \begin{array}{l} 36 : \text{at } 1 : 6 \\ 12 : \text{at } 2 : 0 \\ 12 : \text{at } 1 : 10 \end{array} \right.$	per lb.	Amounts to	$\left\{ \begin{array}{l} 648 \text{ d.} \\ 288 \text{ d.} \\ 264 \text{ d.} \end{array} \right.$
	$60 = \text{the Number of lbs. their Value} = 1200 \text{ d.}$			

Then $60 \text{ lb} : 1200 \text{ d} :: 1 \text{ lb} : 20 \text{ d} = 1 \text{ s. } 8 \text{ d.}$ the Answer required.

Quest. 3. A Vintner Mixeth $31\frac{1}{2}$ Gallons of Malligo Sack, worth 7 s. 6 d. the Gallon; with 18 Gallons of Canary at 5 s. 9 d. the Gallon; $13\frac{1}{2}$ Gallons of Sherry at 5 s. the Gallon; And 27 Gallons of White Wine at 4 s. 3 d. the Gallon. 'Tis required to find what one Gallon of this Mixture is worth.

	Gallons.	s.	d.		Pence.
First	$\left\{ \begin{array}{l} 31\frac{1}{2} : \text{at } 7 : 6 \\ 18 : \text{at } 6 : 9 \\ 13\frac{1}{2} : \text{at } 5 : 0 \\ 27 : \text{at } 4 : 4 \end{array} \right.$	per Gallon	comes to	$\left\{ \begin{array}{l} 2835 \\ 1458 \\ 810 \\ 1377 \end{array} \right.$	
	$90 = \text{the Number of Galls. Their Value} = 6480$				

Then $90 : 6480 :: 1 : 72 \text{ d} = 6 \text{ s.}$ the Rate or Price of one Gallon as was required.

The Proof of all Operations in these sort of Mixtures, is done by comparing the Value of all the Mixture, being sold at the Mean Rate; with the Total Value of all the particular Quantities, supposing they had been sold at their respective Rates unmix'd; if those Sums are equal the work is true.

Sect. 2. Of Alligation Alternate.

Alligation Alternate, is that by which the particular Quantities of every ingredient concern'd in any *Mixture* are found; when the particular Rates of every one of those Ingredients, and the *Mean Rate* are given; being (as it were) the Converse to *Alligation Medial*; as will appear by the following Operations, which admits of three Cases.

Case I. The Particular Rates of any Ingredients proposed to be mixed, and the *Mean Rate* of the whole *Mixture* being given. To find how much of each Ingredient is requisite to compose the *Mixture*; when the whole quantity, or any part thereof is not *Limited*.

Quest. 1. How much *Wheat* at 5 s. the *Buschel*, and *Rye* at 3 s. 6 d. the *Buschel*; will compose a *Mixture* that may be Sold for 4 s. 4 d. the *Buschel*.

Note, In all Questions of this Nature, it will be convenient to place the *Mean Rate* so, as that it may be easily compared with the Particular Rates, in order to find every one of their Differences from the *Mean Rate*, by inspection only.

Thus, the *Mean Rate* = 52 d. $\left\{ \begin{array}{l} \text{Wheat } 60 \text{ d.} \\ \text{Rye } 42 \text{ d.} \end{array} \right.$

Then take the several Differences between the *Mean Rate*, and the particular Rates; Setting down those Differences Alternately, and they will be the quantities required.

Thus 52 $\left\{ \begin{array}{l} 60 \\ 42 \end{array} \right\} \left\{ \begin{array}{l} 10 = 52 - 42 \\ 8 = 60 - 52 \end{array} \right.$

That is $52 - 42 = 10$ for the quantity of *Wheat*.

And $60 - 52 = 8$ for the quantity of *Rye*, that will compose the *Mixture* required.

The Proof by *Alligation Medial*.

Add $\left\{ \begin{array}{l} 10 \text{ Bushels of Wheat at } 60 \text{ d. per Bushel} = 600 \text{ d.} \\ 8 \text{ Bushels of Rye at } 42 \text{ d. per Bushel} = 336 \text{ d.} \\ \hline 18 = \text{The Number of Bushels} = 936 \text{ d.} \end{array} \right.$

Then $18 : 936 :: 1 : 52 \text{ d} = 4 \text{ s. } 4 \text{ d. the Mean Rate.}$

Note, Altho' 10 and 8 do Answer the Question as plainly appears by the Proof; yet they are not the only two Numbers; for this Question and all others of this kind will admit of various Answers, and all in whole Numbers; for any two Numbers that are in the same Proportion to one another, as 10 is to 8 will truly Answer the Question.

Viz. $10 : 8 :: \left\{ \begin{array}{l} 5 : 4 \\ 15 : 10 \\ 20 : 16 \\ 25 : 20 \end{array} \right\} \&c. \text{ ad infinitum.}$

Quest. 2. A Grocer would mix three sorts of Tobacco together, viz. One sort of 18 d. per lb. another Sort of 22 d. per lb. and a third Sort of 2 s. the lb. How much of each Sort must he take, that the whole Mixture may be sold for 20 d. the Pound.

Having set down the given Rates, as before: Then find each of their Differences from the proposed Mean Rate; And place those Differences Alternatly. Thus,

Mean Rate 20 $\left\{ \begin{array}{l} 18 \\ 22 \\ 24 \end{array} \right\} \left\{ \begin{array}{l} 4 + 2 = 24 - 20 \\ 2 = 20 - 18 \\ 2 = 20 - 18 \end{array} \right\}$ and $22 - 20$

These Differences, viz. 6 . 2 . 2 are the quantities Required.

Proof $\left\{ \begin{array}{l} 6 \text{ lb. of Tobacco at } 18 \text{ d. per lb. comes to } 108 \\ 2 \text{ lb. at } 22 \text{ d. the Pound comes to } 44 \\ 2 \text{ lb. at } 24 \text{ d. the Pound comes to } 48 \end{array} \right\} \text{ d.}$
 $10 = \text{the Number of lbs. Their value} = 200 \text{ d.}$

Then 10) 200 (20 the Mean Rate.

Or indeed any three Numbers that have the same Ratio to one another as 6 . & 2 have, will Answer the Question.

That is $6 : 2 :: \left\{ \begin{array}{l} 9 : 3 \\ 12 : 4 \\ 15 : 5 \end{array} \right\} \&c.$

But if only one of the three given Rates, had been greater than the Mean Rate; As suppose 14 d. per lb. 18 per lb. and 24 d. per lb. And the Mean Rate 20 d. as before. Then their differences must have been placed,

Thus 20 $\left\{ \begin{array}{l} 14 \\ 18 \\ 24 \end{array} \right\} \left\{ \begin{array}{l} 4 \\ 4 \\ 6 + 2 \end{array} \right\} \&c. \text{ As before.}$

Quest. 3. A Vintner would make a Mixture, of Malago worth 7 s. 6 d. per Gallon, with Canary at 6 s. 9 d. per Gallon. Sherry at 5 s. per Gallon, and White-Wine at 4 s. 3 d. per Gallon; What quantity of each sort must he take, that the Mixture may be sold for 6 s. a Gallon.

In all Questions of this kind, wherein it is required to mix four things together, Two of them having their prices Grea-

ter, and Two Lesser than the *Mean Rate*; you must always *Alligate* or *Compare* a *Greater* and *Lesser Price* with the *Mean Price*, setting down their *Differences Alternatly*, as in the *First Example* of this *Sect.*

Thus, *Mean Rate* = 72 d. $\left\{ \begin{array}{l} \text{Malago } 90 \text{ d.} \\ \text{White } 51 \text{ d.} \\ \text{Sherry } 60 \text{ d.} \\ \text{Canary } 81 \text{ d.} \end{array} \right. \left\{ \begin{array}{l} 21 = 72 - 51 \\ 18 = 90 - 72 \\ 9 = 81 - 72 \\ 12 = 72 - 60 \end{array} \right.$

Hence 21 Gallons of *Malago*, 12 of *Canary*, 9 of *Sherry*, and 12 of *White* will compose the *Mixture* required.

Or thus 72 $\left\{ \begin{array}{l} \text{Malago } 90 \text{ d.} \\ \text{Sherry } 60 \text{ d.} \\ \text{Canary } 81 \text{ d.} \\ \text{White } 51 \text{ d.} \end{array} \right. \left\{ \begin{array}{l} 12 \text{ Malago} \\ 18 \text{ Sherry} \\ 21 \text{ Canary} \\ 9 \text{ White} \end{array} \right. \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. \text{will, \&c.}$

Either of these *Mixtures* equally *Answer* the *Question*, which may be easily try'd as before in the *Last*, &c.

Case II. The *Particular Rates* of all the *Ingredients* proposed to be mix'd, the *Mean Rate* of the whole *Mixture*, and any one of the quantities to be mix'd being given. Thence to find how much of every one of the other *Ingredients* is requisite to compose the *Mixture*.

Note, This is usually called *Alligation Partial*.

Quest. 4. How much *Wheat* at 5 s. the *Busshel*, must be mix'd with 12 *Busshel* of *Rye* at 3 s. 6 d. a *Busshel*; That the whole *Mixture* may be sold for 4 s. 4 d. the *Busshel*.

In this *Case* you must set down all the *Particular Rates*, with the *Mean Rate*, and find their *Differences* just as before; without any regard had to the quantity given.

Thus, *Mean Rate* 52 d. $\left\{ \begin{array}{l} \text{Wheat } 60 \text{ d.} \\ \text{Rye } 42 \text{ d.} \end{array} \right. \left\{ \begin{array}{l} 10 \\ 8 \end{array} \right.$

Then $\left\{ \begin{array}{l} \text{As the quantity found by the Differences, of the} \\ \text{same Name with the quantity given: Is to the} \\ \text{quantity given: : So is any of the other quantities} \\ \text{found by the Differences: To the quantity of its} \\ \text{Name.} \end{array} \right.$

Thus 8 : 12 :: 10 : 15. the *Quantity* or *Number* of *Busshels* of *Wheat* required.

Quest. 5. How much *Mallago* at 7 s. 6 d. the *Gallon*, *Sherry* at 5 s. the *Gallon*, and *White-Wine* at 4 s. 3 d. the *Gallon* must be mix't with 18 *Gallons* of *Canary* at 6 s. 9 d. the *Gallon*: That the whole *Mixture* may be sold for 6 s. 1 d. the *Gallon*.

The *Terms* being set down, &c. as before, will stand

Thus, *Mean Rate* 72 d. $\left\{ \begin{array}{l} \text{Malago } 90 \text{ d. } \} \} 21 \\ \text{White } 51 \text{ d. } \} \} 18 \\ \text{Sherry } 60 \text{ d. } \} \} 9 \\ \text{Canary } 81 \text{ d. } \} \} 12 \end{array} \right.$

Then, As 12 : 18 :: $\left\{ \begin{array}{l} 21 : 31\frac{1}{2} \text{ Gallons of Malago} \\ 18 : 27 \text{ Gallons of White} \\ 9 : 13\frac{1}{2} \text{ Gallons of Sherry} \end{array} \right.$

That is $31\frac{1}{2}$ Gallons of Malago, 27 of White-Wine, and $13\frac{1}{2}$ of Sherry, being mix'd with 18 Gallons of Canary will make the *Mixture* required.

Or Thus 72 $\left\{ \begin{array}{l} \text{Malago } 90 \} \} 12 \\ \text{Sherry } 60 \} \} 18 \\ \text{Canary } 81 \} \} 21 \\ \text{White } 51 \} \} 9 \end{array} \right.$

Then, As 21 : 18 :: $\left\{ \begin{array}{l} 12 : 10\frac{6}{7} \text{ The Malago} \\ 18 : 15\frac{3}{4} \text{ The Sherry} \\ 9 : 7\frac{1}{2} \text{ The White-Wine} \end{array} \right. \} \text{ \&c.}$

Gallons $\left\{ \begin{array}{l} 10\frac{6}{7} \text{ at } 90 \text{ d. each} = 925\frac{1}{2} \\ 15\frac{3}{4} \text{ at } 60 \text{ d. each} = 925\frac{1}{2} \\ 7\frac{1}{2} \text{ at } 51 \text{ d. each} = 393\frac{2}{7} \\ 18 \text{ at } 81 \text{ d. each} = 1458 \end{array} \right.$

Sum $51\frac{2}{7}$ *Value* = $3702\frac{1}{2}$

Then $51\frac{2}{7}$ $3702\frac{1}{2}$ (72 d = 6s. the *Mean Rate*.)

Therefore the *quantities* are as truly assigned here, as in the last Work.

Case III. The *Particular Rates* of all the Ingredients proposed to be mix'd; and the *Sum* of all their *Quantities*, with the *Mean Rate* of that *Sum* being given; To find the particular *quantities* of the *Mixture*.

This is called **Alligation Total**, and is thus perform'd. Set down all the *Particular Rates*, with the *Mean Rate*, and their *Differences*, as before: Add together all the *Differences* into one *Sum*;

Then $\left\{ \begin{array}{l} \text{As the Sum of all the Differences : Is to the Sum} \\ \text{of all the quantities given : : So is every particular} \\ \text{Difference : To its particular quantity.} \end{array} \right.$

Question 6. Let it be required to mix *Wheat* at 5 s. the *Bushel*, with *Rye* at 3 s. 6 d. the *Bushel*; So as that the whole quantity may be 27 *Bushels*, to be Sold for 4 s. 4 d. *Bushel*. What quantity of each must be taken to make up the *Mixture*?

Q 2

Mean

$$\begin{array}{rcl} \text{Mean Rate } 52 & \left\{ \begin{array}{l} \text{Wheat } 60 \text{ d} \\ \text{Rye } 42 \text{ d} \end{array} \right\} & \left\{ \begin{array}{l} 10 \\ 8 \end{array} \right\} \\ & & \hline & & 18 = \text{their Sum.} \end{array}$$

$$\text{Then } 18 : 27 :: \left\{ \begin{array}{l} 10 : 15 \\ 8 : 12 \end{array} \right\} \text{ The quantities required.}$$

Question 7. Suppose it were required to mix *Malago* at 7s. 6d. the Gallon, with *Canary* at 6s. 9d. the Gallon; *Sherry* at 5s. the Gallon; And *White-Wine* at 4s. 3d. the Gallon: So as that the whole *Mixture* may be 90 Gallons; to be sold for 6s. the Gallon. How much of each Sort will compose that *Mixture*.

$$\begin{array}{rcl} \text{Mean Rate} = 72 \text{ d.} & \left\{ \begin{array}{l} \text{Malago } 90 \\ \text{White } 51 \\ \text{Canary } 81 \\ \text{Sherry } 60 \end{array} \right\} & \left\{ \begin{array}{l} 21 \\ 18 \\ 9 \\ 12 \end{array} \right\} \\ & & \hline & & 60 = \text{their Sum.} \end{array}$$

$$\text{Then } \left\{ \begin{array}{l} 60 : 90 :: 21 : 31\frac{1}{2} \\ 60 : 90 :: 18 : 27 \\ 60 : 90 :: 9 : 13\frac{1}{2} \\ 60 : 90 :: 12 : 18 \end{array} \right. \begin{array}{l} \text{The Gallons of Malago.} \\ \text{The Gallons of White-Wine.} \\ \text{The Gallons of Sherry.} \\ \text{The Gallons of Canary.} \end{array}$$

$$\begin{array}{rcl} \text{Or thus, } 72 & \left\{ \begin{array}{l} \text{Malago } 90 \\ \text{Sherry } 60 \\ \text{Canary } 81 \\ \text{White } 51 \end{array} \right\} & \left\{ \begin{array}{l} 12 \\ 18 \\ 21 \\ 9 \end{array} \right\} \\ & & \hline & & 60 \text{ their Sum.} \end{array}$$

$$\text{Then } \left\{ \begin{array}{l} 60 : 90 :: 12 : 18 \\ 60 : 90 :: 18 : 27 \\ 60 : 90 :: 21 : 31\frac{1}{2} \\ 60 : 90 :: 9 : 13\frac{1}{2} \end{array} \right. \begin{array}{l} \text{Gallons of Malago.} \\ \text{Gallons of Sherry.} \\ \text{Gallons of Canary.} \\ \text{Gallons of White-Wine.} \end{array}$$

Either of these ways do equally Answer the Question, as may be easily tried by *Alligation Medial*. As before, &c.

Note, The Work of these Proportions may be much shortened (especially when there are many Ingredients to be mix'd) if you observe the same Method as was proposed in the Rule of Fellowship. Page 99. &c.

I have made use of the very same Examples both in *Alligation Medial*, and *Alternate* through out the 3 Cases; Being as I presume, much better than if they had been different ones; because the Learner may (if he consider a little of them) easily perceive, not only the difference between the

two Rules, but also wherein the chief difference of each Case in the *Alternate Rule* depends. &c. Not but that I could have inserted many various *Examples*, as also the manner of composing Medicines, &c. which for brevities sake I have omitted, and refer those that desire to see into that business to Sir *Jonas More's Arithmetick*, wherein he will find it largely handled. And so I shall conclude with *Alligation Alternate*, which altho' it gives true *Answers* to *Questions* of that kind; with some little variety, according as the Ingredients are more or less in *Number*; As appears by the foregoing *Examples*. Yet it will not give all the *Answers* as such *Questions* are capable of, nor perhaps those which suit best with the present occasion; Nor can this Imperfection be remedied by common *Arithmetick*; but by an *Algebraick* way of Arguing it may; whereby all the possible *Answers* to any *Question* may be clearly and easily discovered; As shall be shew'd further on in the Second Part.

CHAP. X.

Of Metals, and their Specifick Gravities &c.

SECT. I. Of Gold and Silver.

Pure Gold free from Mixture of other Metals, usually called *Fine Gold*, is of such a Nature and Purity that it will endure the *Fire* without wasting; altho' it were kept continually Melted: And therefore some of the *Ancient Philosophers* have supposed the *Sun* to be a *Globe* of *Liquid* or *Melted Gold*.

Silver having not the Purity of *Gold*, will not indure the *Fire* like it; Yet *Fine Silver* will wast but a very little by being in the *Fire* any reasonable time; whereas *Copper*, *Tinn*, *Lead*, &c. will not only wast, but may be *calcin'd* or *burnt* to a Powder.

Both *Gold* and *Silver* in their Purity, are so very Flexible or soft (like new *Lead* &c.) that they are not so useful either in *Coin*, or otherwise (except to Beat into *Leaf-Gold*, or *Silver*) as when they are allay'd or mix'd and hard'ned with *Copper*, or *Brass*. And altho' most places differ more or less in the *Quantity* of such *Allay*, yet in *England* it is generally agreed on, that.

Stand-

Standard for Gold.

22 Carraets of Fine Gold, and 2 Carraets of Copper, being Melted together shall be esteemed the true Standard for Gold Coin &c. The French and Spanish Gold being very near of the same Standard.

That is, If any quantity or weight of Fine Gold, be Divided into twenty four equal Parts, and 22 of those Parts be mix'd with 2 of the like Parts of Copper; that Mixture is called Standard Gold.

Whence you may observe, that a Carraet is not any certain quantity or weight, but $\frac{1}{24}$ Part of any quantity or weight; and the Minters and Goldsmiths Divide it into 4 equal Parts, which they call Grains of a Carraet; also they subdivide one of those Grains into Quarters, Halves &c.

Standard for Silver.

Eleven Ounces, and Two-Penny-Weight of fine Silver, and Eighteen-Penny-Weight of Copper being melted together, is esteem'd the true Standard for Silver Coin; called Sterling Silver. And so in Proportion for a greater or lesser Quantity; which is a less Proportion of Alloy for Silver than for Gold.

Note, when either Silver, or Gold, is finer than Standard, it's called Better, if courser it's called Worse, and that betterness, or worseness is reckoned by Carraets and Grains of a Carraet in Gold; And by Penny-Weights in Silver, and is thus discovered; the Goldsmiths, or Refiners &c. do take a small Quantity of such Gold as they intend to Try (which they call making an Assay) and weigh it very exactly, then they put it into a Crucible and melt it in a strong Fire, so long that if there be any Copper, or other Alloy mixt with it, that Alloy may be consum'd or burnt away: when it's cold they weigh it very exactly again, and if it have lost nothing of it's first weight, they conclude it is fine Gold, but if the Loss be $\frac{1}{4}$ Part, they call it 23 Carraets fine, or one Carraet Better than Standard: if it have lost $\frac{2}{4}$ Parts, it's 22 Carraets fine, or Standard. If $\frac{3}{4}$ Parts, it's said to be 21 Carraets fine, or rather one Carraet worse than Standard, and so in Proportion as it happens to be Better, or Worse.

In the same manner they make their Assay on Silver, only they compute it's Loss by Penny-Weights, &c.

The Author of the Present State of England mention'd before (Page 32) says.

‘ That

‘ That the *English Coin* may not want neither the Purity nor Weight required, it is most wisely and carefully provided, that once every Year the *Chief Officers* of the *Mint* appear before the *Lords* of the *Council* in the *Star-Chamber* at *Westminster*, with some pieces of all sorts of *Moneys* Coined the foregoing Year, taken at *Adventure* out of the *Mint*; and kept under several Locks by several Persons ‘till that appearance, and then by a *Jury* of 24 able *Goldsmiths* in the presence of the said *Lords*; every piece is most exactly Weighed and Assay’d.

This if it were constantly practised must need keep our *Coin* to its true *Standard* &c;

Many pretty *Questions* may be started concerning the *Fines* of *Gold* and *Silver* &c.

Example 1.

If an *Ingot* of *Silver* weighing 787 $\frac{3}{4}$. 14 pwt. 6 Grains Be 11 oz. 6 Penny-Weight fine. How much fine *Silver* is there in it, and what amounts it to, at 5 s. 1 $\frac{1}{2}$ d. the *Ounce*.

This *Ingot* is *Better* than *Standard* by 4 pwt. For 11 oz. 2 pwt. = 222 pwt. the fine *Silver* in 12 oz. of *Standard*. But 11 oz. 6 pwt. = 226 pwt. the fine *Silver* in 12 oz. according to the *Question*.

First 787 oz. 14 pwt. 6 Grains = 378102 Grains.

And 12 oz. = 240 pwt.

Then As 240 : 226 :: 378102 : 356046 $\frac{2}{5}$ = 741 oz. 15 pwt. 6 $\frac{1}{5}$ Grains; the fine *Silver* in that *Ingot*.

Which at 5 s. 1 $\frac{1}{2}$ d. the *Ounce*, amounts to 190 l. 1 s. 6 d. and near a *Half-Penny*.

Example 2.

If an *Ingot* of *Gold* weighing 115 oz. 13 pwt. 18 Grains; Be $\frac{1}{4}$ of a *Grain* worse than *Standard*. How much *Standard Gold* is there in it, and what comes it to, at 3 l. 11 s. an *Ounce*.

First 115 oz. 13 pwt. 18 gra. = 55530 Grains *Troy*.

Then 24) 55530 (2313,75 = a *Carract* of that quantity

And 4) 2313,75 (578,4375 = a *Grain* of that *Carract*.

Consequently 4) 578,4375 (144,609375 = $\frac{1}{4}$ of a *Grain*.

Again, 2313,75 $\times$ 22 = 50902,5 ought to be the fine *Gold* in that *Ingot*, if it had been *Standard*.

But

But $50902,5 - 144,609375 = 50757,890625$ is the quantity of fine Gold according to the Question.

Therefore $50902,5 : 50757,890625 :: 55530 : 55372,244 \&c.$ Grains $= 115 \text{ oz. } 7 \text{ pwt. } 4,244 \&c.$ Grains Troy, being the quantity of Standard Gold in that Ingot. As was required.

Next for the Value of it, at $3 \text{ l. } 11 \text{ s. per Ounce}$; $1 \text{ oz.} = 480 \text{ Grains.}$ And $3 \text{ l. } 11 \text{ s.} = 71 \text{ s.}$

Consequently $480 : 71 :: 55372,244 \&c : 8190,4777 \&c.$ $= 409 \text{ l. } 10 \text{ s. } 5\frac{1}{4} \text{ d.}$ very near; Being the Value of that Ingot. As was required.

Or the last Question may be otherwise wrought thus, $115 \text{ oz. } 13 \text{ pwt. } 18 \text{ Grains} = 115,6875.$ And $\frac{1}{4}$ of a Grain, of a Carraet is $\frac{1}{8}$ (viz. the $\frac{1}{4}$ of $\frac{1}{4}$)

Then $22 - \frac{1}{8} = 21\frac{7}{8} = 21,9375.$

Consequently $22 : 21,9375 :: 115,6875 : 115,358842 \&c.$ $= 115 \text{ oz. } 7 \text{ pwt. } 4,244 \text{ Grains} \&c.$ As before.

Next for the Value, As $1 : 3,55 :: 115,358842 : 409,523889$ $= 409 \text{ l. } 10 \text{ s. } 5\frac{1}{4} \text{ d.}$ very near. As before.

Sect. 2. The Specifick Gravity of Metals, &c.

I take an enquiry made about the different Gravities or Weights of Metals and other Bodies, to be (not only a work of Curiosity, but also) of very good use upon many Occasions. Therefore several Authors have given us such Proportions, or difference of their Weights, as they are said to have one to another: Supposing every one of them to be of the same Magnitude or Bigness. Some of which I shall here insert.

1. Henry Van Etten, in his *Mathematical Recreations* Printed Anno 1633; Sets down the Proportion of their Weights. Thus.

Gold 1875 . Lead 1165 . Silver 1040 . Copper 910 . Iron 810 . Tin 750 . Water 100.

2. One Alsted, in his *Encyclopædia*; Printed 1649 hath them Thus.

Gold 1875 . Quick-Silver 1500 . Lead 1165 . Silver 1040 . Copper 910 . Iron 806 . Tin 750 . Honey 150 . Water 100 . Oyl 90. These seem to be taken from these of Van Etten's, with some Additions only.

3. The Ingenious Mr. Oughtred, in his *Circles of Proportions*, Printed Anno 1660 hath their Proportions according to the Experiments of one Marinus Ghetaldi, in his Treatise called *Archimedes Promotus*. Thus.

Gold 3990 . Quick-Silver 2850 . Lead 2415 . Silver 2170 . Brass . 1890 . Iron 1680 . Tin 1554. 4. la

4. In the Philosophical Transactions, Number 169 and 199. There is an Account of a great many Experiments of this kind; from whence I collected these following. *Viz.* Gold 18888 . Mercury 14019 . Lead . 11345 . Silver 11087 . Copper 8843 . Hammer'd Brass 8349 . Cast Brass 8100 . Steel 7852 . Iron 7643 . Tin 7321 . Pump-water 1000 . These last Proportions being approved of and published by Order of the Royal Society; seem to be unquestionably true: Nevertheless, because they differ so much from the before mention'd (and those from one another) I have for my own satisfaction made several Experiments of that kind: And have (*I presume*) obtained the Proportions of *Weight* that one body bears to another, of the same bulk or magnitude; as nicely as the Nature of such matter, as may be contracted or brought into a lesser body (*viz.* either by Drying, Hammering or otherwise) will admit of; which are as followeth.

		Ounces Troy	Ounces Averdupois
Cubick Inch of	Fine Gold is,	10,359273	= 11,365602
	Standard Gold,	9,962625	= 10,930422
	Quick-Silver,	7,384411	= 8,101753
	Lead,	5,984010	= 6,553885
	Fine Silver,	5,850035	= 6,418324
	Standard Silver,	5,556769	= 6,096569
	Rose Copper,	4,747121	= 5,208369
	Plate Brass,	4,404273	= 4,832116
	Cast Brass,	4,272409	= 4,630300
	Steel,	4,142127	= 4,544505
	Common Iron,	4,031361	= 4,422979
	Block Tin,	3,861519	= 4,236638
	Fine Marble,	1,429411	= 1,568859
	Common Glass,	1,360841	= 1,493037
	Alabaster,	0,988456	= 1,084477
	Dry Ivory,	0,962083	= 1,055542
	Dry Box-wood,	0,543282	= 0,596057
	Sea Water,	0,542742	= 0,594894
	Common Clear Water,	0,527458	= 0,578697
	Red Wine,	0,523766	= 0,574646
	Proof Spirits or Brandy,	0,489268	= 0,536796
	Sound drie Oak,	0,489008	= 0,536569
	Lint-seed Oyl,	0,491591	= 0,539345
	Oyl Olive,	0,481569	= 0,528350

R

In

In this *Table* you have the *Specifick Gravity* or *Weight* of a *Cubick Inch*, of various sorts of *Bodies*, both in *Troy Ounces* and *Averdupois Ounces* and *Decimal Parts* of an *Ounce*. which I can assure you required more Charge, Care and Trouble, to find out nicely, than I was at first aware of.

Now from hence it will be easie to determin the *Weight* of any proposed *Quantity*, of the same matter or kind with those in the *Table*; its *Solid Content* being given in *Cubick Inches*. For it is plain, that if the *Number* of *Cubick Inches* contain'd in any given *Quantity*, be *Multiplied* with the *Tabular Weight* of one *Inch* (of the same kind of matter) the *Product* will be the *Weight* of that *Quantity* in *Ounces* &c.

Example.

Suppose it were required to find the *Weight* of a piece of *Marble*, containing three *Solid Feet* and 40 *Cubick Inches*.

First $1728 \times 3 = 5184$ the *Cubick Inches* in 3 *Solid Feet*.

And $5184 + 40 = 5224$ the *Number* of *Cubick Inches* in the piece of *Marble*.

Then $5224 \times 1,429411 = 7410,066624$ *Ounces Troy*.

Or $5224 \times 1,568859 = 8195,719416$ *Ounces Averdupois*.

The *Weight* of that piece of *Marble*, in *Ounces*, &c. which is easily brought into *Pounds* &c. The like for any of the rest.

The converse of this Work is as easie; viz. if the *Weight* of any proposed *Quantity* be given, thence to find the *Solid Content* of that *Quantity* in *Cubick Inches* &c. Thus Divide the given *Weight* of the Proposed *Quantity* (it being first *Reduced* into *Ounces*, &c.) by the *Tabular Weight* of one *Inch* (of the same kind of Matter) and the *Quotient* will be the *Number* of *Cubick Inches* contained in that *Quantity*.

Note. If you would find what *Weight* any *Quantity* of those *Bodies* mentioned in the *Table*, will have when is immersed or put into *Water*, you must *Subtract* the *Weight* of an equal *Quantity* of *Water* (with that of the *Body*) from the *Weight* of the proposed *Body* (if it be heavier than *Water*) and there will *Remain* the *Weight* required. As for Instance,

A *Cubick Inch* of *Lead* = 5,984010 } *Ounces Troy* &c.

A *Cubick Inch* of *Water* = 0,542742 }

their difference is, = 5,441268 the *Weight* of a *Cubick Inch* of *Lead* in the *Water*, &c.

C H A P. XI.

Evolution, or Extracting the Roots out of all Single Powers, by one General Method.

SECT. I.

Evolution is the *Unravilling* or as it were the *Unfolding* and *Resolving* any proposed *Power* or *Number*, into the same parts of which it was composed, or supposed to be made up of. Now in order to perform that, it will be convenient to Consider how those *Powers* are Composed. &c.

A *Square Number* is that which is *equally equal*; Or which is contained under to *equal Numbers*. Euclid 7. Def. 18.

Thus the *Square Number* 4 is composed of the two *equal Numbers* 2 and 2. viz. $2 \times 2 = 4$.

Or the *Square Number* 9 is composed of the two *equal Numbers* 3 and 3. viz. $3 \times 3 = 9$. According to Euclid.

That is, if any *Number* be *Multiplied* into it self; that *Product* is called a *Square Number*.

A *Cube* is that *Number* which is *equally equally Equal*, or which is contained under three *equal Numbers*. Eu. 7. Def. 19.

Thus the *Cube Number* 8 is composed of the three *equal Numbers* 2 and 2 and 2. viz. $2 \times 2 \times 2 = 8$. &c.

That is, if any *Number* be *Multiplied* into it self, and that *Product* be *Multiplied* with the same *Number*; the second *Product* is called a *Cube Number*.

These Two, viz. the *Square*, and *Cube Numbers* borrow their Names from *Geometrical Extentions* or *Figures*; as from the three *Signal Quantities* mentioned in Page 2

That is, a *Root* is represented by a *Line* or *Side*, having but one *Dimention*, viz. that of *Length* only.

The *Square* is a *Plain* or *Figure* of Two *Dimentions*, having *Equal Length* and *Breadth*. The *Cube* is a *Solid Body* of Three *Dimentions*; having *Equal Length*, *Breadth*, and *Thicknes*: But beyond these Three, *Nature* proceeds not, as to *Local Extention*. That is, the *Nature* of *Place* or *Space*, admits no room for other ways of *Extention*, than *Length*, *Breadth* and *Thicknes*. Neither is it possible to form, or compose any *Figure* or *body* beyond that of a *solid*.

And therefore all the *Superior Powers* above the *Cube* or *Third Power*; As the *Biquadrat* or *Fourth Power*, the *Solid* or *Fifth Power* &c. are best explain'd and understood by a *Rank* or *Series* of *Numbers* in *Geometrical Proportion*.

For Instance.

Suppose any *Rank* of *Geometrical Proportionals*, whose *First Term* and *Ratio* are the same; And to them let there be

R 2

assigned

Assigned a Series of Numbers in *Arithmetical Progression*, beginning with an *Unit* or 1. whose common difference is also 1. as in Page 79.

Thus $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 8 & 16 & 32 & 64 & 128 \end{matrix}$ Indices. &c. in $\div$

Then are those Numbers in $\div$ produced by a continued Multiplication of the First Term or Root into it Self; And those in *Arithmetical Progression* or Indices, do shew what Degree or Power each Term in the Geometrical Proportion is of.

For Example; In this Series of $\div$ 2 is both the First Term or Root and common Ratio of the Series.

Then $2 \times 2 = 4$ the Second Term or Square.

And $2 \times 2 \times 2 = 8$ Or $4 \times 2 = 8$ the Cube Or Third Term.

Again $2 \times 2 \times 2 \times 2 = 16$ Or $8 \times 2 = 16$ the Fourth Term or Biquadrat. And so on for the rest.

Note, This is called *Involution*, viz. When any Number is drawn into it self, and afterwards into that Product, &c. its said to be so often involved into it self; And the Indices are the Exponents of their respective Powers so involved.

And according to these *Involutions*, is formed the following Table of Powers; wherein the Root is only one Single Figure.

Root, or single Side.	Square, or 2d. Power.	Cube, or the third Power.	Biquadrat, or Square Squared; being the fourth Power.	Sur-solid, or the fifth Power.	Square Cubed, or Cube Squared; the sixth Power.	The Second Sur-solid, or seventh Power.	The Biquadrat Squared, or the eighth Power.	The Cube Cubed, or the ninth Power, &c.
Index (2)	Index (3)	Index (4)	Index (5)	Index (6)	Index (7)	Index (8)	Index (9)	Index (9)
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

This Table plainly shews (by Inspection) any Power (under the Tenth) of all the Nine Figures; and from thence may be taken

taken the nearest *Root* of any *Square*; *Cube*; *Biquadrate*, &c. of any *Number* whose *Root* or *Side* is a *Single Figure*.

But if the *Root* consist of *two*, *three*, or more places of *Figures*, then it must be found by piece Meal, or *Figure* after *Figure*.

The *Extraction* of all *Roots*, above the *Square* (*viz.* of the *Cube*; *Biquadrat*; *Sur-solid*, &c.) hath heretofore been a very tedious and troublesome piece of *Work*; All which is now very much shortned, and rendred *Easie*, as will appear further on.

When any *Number* is proposed to have its *Root* *Extracted*, the first *Work* is to prepare it, by *Points* set over (or under) their proper *Figures*; according as the given *Power*, whose *Root* is sought doth require; And that's done by considering the *Index* of the given *Power*, which for the *Square* is 2. for the *Cube* is 3. for the *Biquadrat* is 4. &c. (as in the precedent *Table*) Then allow so many places of *Figures* in the given *Power*, for each *Single Figure* of the *Root*, as its *Index* denotes; always beginning those *Points* over the place of *Unity*, and ascend towards the *Left Hand* if the given *Number* be *Integers*, and descend towards the *Right Hand* in *Decimal parts*. As in these following.

Suppose any given *Number*; As 75640387246 which I will a long hereafter call the *Resolvend*.

Then if it be required to *Extract* any of the following *Roots*, it must be pointed (according to the forementioned *Consideration*) in this manner.

Viz. For the	{	Square Root Thus	75640387246
		Cube Root	75640387246
		Biquadrat Root	75640387246
		Sur-solid Root	75640387246

Or suppose the *Number* to be 0,674035982

Then for the	{	Square Root Thus	0,6740359820
		Cube Root	0,674035982
		Biquadrat	0,674035982000

Now the *Reason* of *Pointing* the given *Resolvend* in this manner; *viz.* the allowing *Two Figures* in the *Square*; *Three Figure*

Figures in the Cube, and Four Figures in the Baquadrat, &c. For one *Figure in the Root*, may be made Evident several ways; but I think its easily conceiv'd from the *Table of Single Powers*, wherein you may observe that all the *Powers* of the *Figure 9.* (which is but a *Single Figure*) have the same Number of places of *Figures*, as the *Index* of those *Powers* denotes: Therefore so many places of *Figures* must be taken or assigned for every *Single Figure* in the *Root*. Consequently by these *Points* is known how many places of *Figures* there will be in the *Root*, viz. So many *Points* as there are, so many *Figures* there must be in the *Root*, and whether they must be *Integers*, or *Dicimals parts*, is easily determined by the respective places of the *Points*.

SECT. 2. To Extract the Square Root.

And First how to Extract the Square Root, according to the common Method.

Having pointed the given *Resolvend* into *Periods* of *Two Figures*, as before directed; then by the *Table of Powers* (or otherwise) find the greatest *Square* that is contained in the first *Period* towards the *Left Hand*; (setting down its *Root*, like a *Quotient Figure* in *Division*), and *Subtract* that *Square* out of the said *Period* of the *Resolvend*: To the *Remainder* bring down the next *Period* of *Figures*, for a *Dividend*, and double the *Root* of the first *Square* for a *Divisor*: inquiring how oft it may be had in that *Dividend*; So as when the *Quotient Figure* is annexed to the *Divisor*, and that *increased Divisor* being *Multiplied* with the same *Quotient Figure*, the *Product* may be the greatest *Number* that can be taken out of that *Dividend*; which *Subtract* from the said *Dividend*, and to the *Remainder* bring down the next *Period* of *Figures*, for another *New Dividend*; Then see how oft the *Last increased Divisor*, can be had in the *New Dividend* (with the same *Caution* as before, viz.) So as that the *Quotient Figure* being *Annexed* to the *Divisor*, and that *increased Divisor* *Multiplied* with the same *Quotient Figure*, their *Product* may be the greatest *Number* that can be *Subtracted* from the *New Dividend*. (As before) And so proceed on from *Period* to *Period*; (viz. from *Point* to *Point*) in the very same manner, until all be finished.

An *Example* or *Two* being well observed, will render the *Work* of forming the *New Divisors*, &c. more *Plain* and *Easie*, than can be *Expressed* in a *Multitude* of *Words*.

Example

Example 1. Let it be required to *Extract the Square Root* out of 572199960721. This *Resolvend* being prepared or pointed as before directed will stand;

Thus 572199960721 (756439 the *Root*
49 = the greatest *Square* in 57.

1. Divisor, 145) 821
5 725 = 145 × 5
2. Divisor 1506) 9699
6 9036 = 1506 × 6
3. Divisor 15124) 66396
4 60496 = 15124 × 4
4. Divisor 151283) 590007
3 453849 = 151283 × 3
5. Divisor 1512869) 13615821
9 13615821 = 1512869 × 9

Proof 756439 × 756439 = 572199960721 the *Resolvend*.

Example 2. What's the *Square Root* of 1850701,764025

Operation 1850701,764025 (1360,405

$$\begin{array}{r}
 23) \overline{185} \\
 \underline{3} \quad 69 \\
 266 \quad 1607 \\
 \underline{6} \quad 1596 \\
 17204) \overline{1101,76} \\
 \underline{4} \quad 1088 \quad 16 \\
 1720805) \overline{13 \ 604025} \\
 \underline{5} \quad 13 \ 604025 \\
 (0)
 \end{array}$$

Hence 1360,405 is
the *Root* required.

Example 3. What's the *Square Root* of 0,06076225 *De-*
cimal parts.

Operation 0,06076225 (0,2465 the *Root* required.

$$\begin{array}{r}
 ,44) \overline{,04} = ,2 \times ,2 \\
 \underline{4} \quad 176 \\
 ,486 \quad 3162 \\
 \underline{6} \quad 2916 \\
 ,4925) \overline{24625} \\
 \underline{5} \quad 24625 \\
 (0)
 \end{array}$$

Proof $\begin{cases} 0,2465 \times 0,2465 = \\ 0,06076225 \end{cases}$ the
Resolvend.

What

What is here done in Whole Numbers, Mix'd Numbers, and Decimals, may also be done in Vulgar Fractions; if you first Change the given Fraction into Decimals. (As in Sect. 5. Page 68.)

Example 4. Let it be required to Extract the Square Root of $\frac{16}{25}$. First $\frac{16}{25} = 0,64$

Then 0,64 (,8 the Root required.

$$\begin{array}{r} .64 \\ (0) \end{array}$$

In these Four Examples the Resolvend hath been a perfect Square; and therefore the Root hath been Extracted without Leaving any Remainder: But it very often happens that the Resolvend is not a true Figurate Number, according to the proposed Power. That is, it's not a perfect Square, Cube, Biquadrate, &c. And then something will Remain after the Extraction hath been made throughout all the Points. Such Numbers are called **Surd Numbers**, and their Roots can never be truly found, but will become a Continued Series *ad infinitum*; If to the Remainder there be still annexed Cyphers according as the proposed Power requires, viz. by Two's in the Square; Three's in the Cube; Four's in the Biquadrate, &c. And the Operations continued on as before.

Example 5. Suppose it were required to Extract the Square Root of 6968

Operation

$$\begin{array}{r} 6968 \quad (83,4745. \text{ \&c.} \\ 64 \\ \hline 163) \quad 568 \\ \underline{3} \quad 489 \\ 1664) \quad 79,00 \\ \underline{4} \quad 66 \ 56 \\ 16687) \quad 12 \ 4400 \\ \underline{7} \quad 11 \ 6809 \\ 166944) \quad 759100 \\ \underline{4} \quad 667776 \\ 1669485) \quad 9132400 \\ \underline{5} \quad 8347425 \\ 1669490 \quad 784975 \text{ \&c.} \end{array}$$

Thus the Root of a **Surd Number** may be continued on to what Exactness you please, but cannot be truly found.

In my *Compendium of Algebra*, Chap. 9. I have proposed another way of Extracting the Square Root, and there given Examples of the Work: Which to Avoid prolixity is thus,

Having

Having pointed the given *Resolvend*, and taken the *Greatest Square* to the *First Point* from it, as before. Then *Divide* the *Remainder* of the whole *Resolvend* by 2 (that is *halve* it) and *Point* it a New. (This I call a *New Dividend*) Then make the *Root* of the *First Square* a *Divisor*, inquiring how oft it may be found in the *New Dividend* to the next *Figure* forward, reserving that *Figure* under the next *Point*, for the *half Square* of the *Quotient Figure*. Which being found, *Multiply* the *Divisor* with it, adding to that *Product* the *Tens* of the *half Square* if there be any; As in plain *Division*.

Then annex the *Quotient Figure* to the *Last Divisor*, for a *New Divisor*, with which proceed in all *Respects* as with the *Last Divisor*; And so on until all be finished.

Example 6. What's the *Square Root* of 2990667969

$$\begin{array}{r}
 \text{Operation} \quad 2990667969 \\
 - \quad 25 \quad \quad \quad (5 \text{ The First Single Root.}) \\
 2) \quad 490667969 \quad \text{The Remainder to be Divided by 2} \\
 \\
 \text{First Root } 5) \quad 245333984,5 \quad (54687 \\
 + \quad 4 \quad 208 = 5 \times 4 : + \frac{1}{2} \text{ the Square of 4 viz. } \frac{1}{2} \times 4 = 8 \\
 \text{Divisor } 54) \quad 3733 \\
 + \quad 6 \quad 3258 = 54 \times 6 : + \frac{1}{2} \text{ the Square of 6} \\
 \text{Divisor } 546) \quad 47539 \\
 + \quad 8 \quad 43712 = 546 \times 8 : + \frac{1}{2} \text{ the Square of 8} \\
 \text{Divisor } 5468) \quad 382784,5 \\
 + \quad 7 \quad 382784,5 = 5468 \times 7 : + \frac{1}{2} \text{ the Square of 7} \\
 \quad \quad \quad (0)
 \end{array}$$

Hence the *Root* is found to be 54687 As was required.

All the *difficulty* in this *Method*, is only in the true placing of the *half Square* of the *Quotient Figure*, when it happens to be an *odd Number*; In that *Case* you must bring down one *Figure* more of the *Dividend*; viz. of the next *period*; under which place the *odd 5* that will always arise from the *half Square* of an *odd Number*: As 7 whose *Square* is 49 the *half* of which is 24,5 to be placed as in the *Last operation* of this *Example*.

N. B. When the *Number* of *Figures* in the *Root* of a *Surd Number* are *limited*; you need not proceed in *Extracting* the *whole Root* as before; but only to one *Figure* more than half the *designed Number* of *Figures*; for the rest may be obtained by plain *Division*.

S

Example

Example 7. Suppose it were required to *Extract* the *Square Root* of 7 (a *Surd Number*) to have *Twelve* places of *Figures* in it.

7	(2,645751	First part of the Root.
4		
3		
2)	1,50 = half the Remainder.	
+ .6	1,38 = 2 × .6 : + ½ the Square of 0,6 = 0,18	
2,6)	1200	
+ .04	1048	
2,64)	152000	
+ .005	132125	
2,645)	1987500	
+ .0007	1851745	
2,6457)	13575500	
+ .00005	13228625	
2,64575)	34687500	
+ .000001	26457505	
2,645751	8229995	

Having thus got 7 of the 12 *Figures* required in the *Root*; the rest may be easily found by the *Contracted* way of *Division* proposed in *Page 68*.

Thus 2,645751) 8229995 (2,64575131106

7937253

292742

264575

28167

26457

1710

1697

(13)

Hence I find the *Root* of 7 To be 2,6457131106 As was required.

Thus you have *Two* ways of *Extracting* the *Square Root* either of them may be *practised* as every one likes best.

Sect. 3. To *Extract* the *Cube Root*.

The *Method* that I shall here propose for *Extracting* the *Cube Root*, admits of *Two Cases*; both which are to be very well observed.

Having pointed the given *Resolvend*, (as before directed) viz. into *Periods* of *Three Figures*; Then seek a *Cube Number* by the *Table of Powers* (or otherwise) that comes the nearest to the *First Period* of the *Resolvend*, whether it be *Greater* or *Less* than that *Period*.

Case 1. If the *Cube Number* so taken, be *Less* than the *First Period* of the *Resolvend*.

Call its Root *Less than Just*.

And *Subtract* that *Cube* from the *First Period* of the *Resolvend*.

Case 2. But if that *Cube* be greater than the *First Period* of the *Resolvend*.

Call its Root *More than Just*.

And *Subtract* the *Resolvend* from that *Cube*, annexing *Cyphers* to it, that so *Subtraction* may be made.

To the *First Root* whether it be *Less*, or *More* than *Just*, annex so many *Cyphers* as there are remaining *Points* over the whole *Numbers* of the *Resolvend*; and *Multiply* it with 3. Then make that *Product* a *Divisor*; by which you must divide the *Difference* between the *Resolvend* and the foresaid *Cube*, then will that *Quotient* be the *Resolvend* depressed to a *Square*; and therefore it must be pointed as such: viz. into *Periods* of *Two Figures* each. That being done, make the *First Root* (without those *Cyphers* that were annex'd to it) a *Divisor*, Inquiring how oft it may be found in the *First Period* of the *New Resolvend*, (as before in *Extracting* the *Square Root*) with this Consideration, that if the *Root*, (now a *Divisor*) be *Less* than *Just*, as in *Case 1.* you must annex the *Quotient Figure* to it, and then *Multiply* the *Root* so increased, into the said *Quotient Figure*; Setting down the *Units* place of their *Product* under the pointed *Figure* of that *Period*, *Subtracting* it, as in *Division*. And so on from one *Period* to another. As before.

But if the said *Root* (now a *Divisor*) be *More* than *Just*, as in *Case 2.* Then you must *Subtract* the *Quotient Figure* from a *Cypher* annexed, or suppose to be annexed to the said *Divisor*; *Multiplying* the *Root* so decreased into the *Quotient Figure*; setting down their *Product* as before, &c. An Example or two in each *Case* will render the *Work Plain* and *Easie*.

Example 1.

What's the *Cube Root* of 146363183 the given *Resolvend*, to be pointed thus 146363183 (5 the *First Root*, Less than just.

125 = the nearest *Cube* to 146

500 × 3 = 1500) 21363183 (14242,12 New *Resolvend*

First Root 5) 14242,12 (527 the *Root* required.

+ 2 104

1. *Divisor* 52) 3842

+ 7 3689

2. *Divisor* 527 (153) the *Remainder* to be rejected.

Here the *Root* 527 is the *true Root* at the *first Operation*, as may be easily tried by *involving* it.

That is $527 \times 527 \times 527 = 146363183$ the given *Resolvend*. But if it had not been the *true Root*; Then every thing that hath been here done must have been repeated; Only instead of the *first Single Root* (viz. 5.) you must have taken the *increased Root* (viz. 527) and this I call a *Second Operation*; Which would increase the *Last Root* to Nine places of *Figures*. viz. every *Operation* Triples the Number of places in the *Last Root*; As will appear further on.

N. B. It often happens that Four, or sometimes Five Places of *Figures* may be taken into the *Root*; Especially when the *Second Place* proves to be a *Cypher*. That is, when the *First Cube* comes very near to the *First Period* of the *Resolvend*.

Example 2.

What's the *Cube Root* of 67507824239 (4000 *Root* Less than just)

First nearest *Cube* = 64

Root 4000 × 3 = 12000) 3507824239 (292318,68

+ 4)

1. *Divisor* 40) 2923

+ 7 2849

2. *Divisor* 407) 7418

+ 1 4071

3. *Divisor* 4071) 3347,68

+ 8 3257,44

Root = 4071,8 &c.

In this *Example* I have taken *Five Figures* into the *Root* because the *Second Place* proved to be a *Cypher*. And in the

Thus	99207)	623377841921091	(62836,45
	— 6	5952384	
Divisor	992064	28139441	
	— 2	19841276	
Divisor	9920638	829816592	
	— 8	793650976	
(*)	99206372	3616561610	
	— 3	2976191151	
	992063717	64037045991	
	— 6	59523822984	
	9920637164	451322300700	
	— 4	396825486544	
	9920637163,6	5449681415600	
	— 05	4960318581775	
	9920637163,55	&c.	

Note in this Operation all the Figures on this side the Line, and all the new Divisors after the (*) are useless, and might have been Omitted.

Last Root 9920700000

— 62836,45 &c.

9920637163,55 the Root required.

Thus I have obtained the Cube Root to Twelve Places of Figures. viz. 9920637163,55, at Two Operations; being but an Unit too much in the Last Place of it, as may be tried by involving it to a Cube, and comparing that Cube with the given Resolvend.

In the same manner the Cube Roots of Decimal Parts; or of Vulgar Fractions, being first Changed into Decimals, may be Extracted.

Sect. 4. To Extract the Biquadrat Root.

In Extracting the Biquadrat Root, or that of the Fourth Power; (and indeed the Roots of all even Powers) there is some small Difficulties, not so easily express'd and Explain'd in a few words, as they are by an Algebraick Theorem (such as shall be shewed further on) I have therefore in this place, made Choice of Extracting such Roots by Two several Extractions; And the rather because I presume the Reader by this time thoroughly acquainted with the business of Extracting the Square Root, by which this may easily be performed. Thus

First Extract the Square Root, of the proposed Resolvend. Then the Square Root of that first Root will be the Biquadrat Root required.

Exam

Example 1. What's the *Biquadrat Root* of 4857532416.
First Extract its *Square Root*.

Thus $\begin{array}{r} 4857532416 \\ - 36 \\ \hline 1257532416 \end{array}$ the greater *Square*, whose *Root* is 6
1257532416 *Remainder* to be *Divided* by 2

First Root 6) $\begin{array}{r} 628766208 \\ + 9 \\ \hline 69 \\ 4826 \\ + 6 \\ \hline 696 \\ 668620 \\ + 9 \\ \hline 6969 \\ 626805 \\ 418158 \\ 418158 \\ \hline (0) \end{array}$ (69696

Then $\begin{array}{r} 69696 \\ - 4 \\ \hline 29696 \end{array}$ { being the 1st. Root, whose *Square*
Root must now be *Extracted*.
29696 *Remainder* to be *Divided* by 2

1st. Root 2) $\begin{array}{r} 14848 \\ + 6 \\ \hline 26 \\ 138 \\ 1048 \\ + 4 \\ \hline 264 \\ 1048 \\ \hline (0) \end{array}$ (264 the *Biquadrat Root* as was
Required

This is so *Easie* I need not *Insert* any more *Examples*.

Sect. 5. To Extract the *Sur-solid Root*.

Having pointed the given *Resolvend* according as its *Index* Denotes; viz. into *Periods* of *Five Figures*; Seeking such a *Sur-solid Number* in the *Table of Powers* (or otherwise) as comes the nearest to the *First Period* of the *Resolvend*, whether *Greater* or *Less*; and call its respective *Root* accordingly; viz. *More* than *Just*; Or *Less* than *Just*; annexing so many *Cyphers* to it, as there are remaining *Periods* of whole *Numbers* in the *Resolvend*. As before in *Extracting* the *Cube Root*.

Then find the *difference* between the *Resolvend*; and the *Sur-solid Number* so taken, by *Subtracting* the *Lesser* from the *Greater*, (as before in the *Cube*) Next find the *Cube* of the fore-said *Sur-solid Root* with its annexed *Cyphers*, (which you

you may also do by the *Table of Powers*) and *Multiply* that *Cube* with 5 the *Index* of the *Sur-solid*, the *Product* must be a *Divisor*, by which the *Difference* between the *Resolvend*, and the *Sur-solid Number* must be *Divided*, that so it may be *depressed* to a *Square* (as before in the *Cube*) which must be *pointed* into *Periods* of *Two Figures* each, calling it the *New Resolvend*. (As before) Then make the *First Root*, without its *Cyphers*, a *Divisor Inquiring*, how oft it may be found in the *First Period* of the *New Resolvend*, with this *Consideration*, if the *Root* (now a *Divisor*) be *Less* than *Just*, you must *annex Twice* the *Quotient Figure* to it; but if it be *More* than *Just*, you must *Subtract Twice* the *Quotient Figure* from a *Cypher* either *annexed*, or suppose to be *annex'd* to that *Divisor* or *Root*, *Multiplying* it so *increased*; or *diminished* with the said *Quotient Figure*, setting down their *Product*, &c. As before, An *Example* in each *Case* will render it plain and *Easie*.

Example 1. Suppose it be required to *Extract* the *Sur-solid Root* out of this *Number* 12309502009375

12309502009375 The *Resolvend Pointed*.

The nearest *Sur-solid Number* to 1230 the *First Period* of the *Resolvend*, is 1024 whose *Root* is 4 being *Less* than *Just*,

Therefore 12309502009375
— 1024

2069502009375 their *Difference*.

Next the *Cube* of 400 is 64000000 per *Table*, &c.

And $64000000 \times 5 = 320000000$ the *Divisor*.

Then $320000000 \mid 2069502009375$ (6467 &c.

First Root 4 6467 (15

— $1 \times 2 = 2$ 42 *First Root* = 400

Divisor 42 2267 — 15

— $5 \times 2 = 10$ 2150 the true *Root* 415 as requir'd.

Divisor 430 (117) the *Remainder* to be rejected.

That is 415 is the *Sur-solid Root* of the given *Resolvend*. As may be easily tried by involving it to the *Fifth Power*.

Viv. $415 \times 415 \times 415 \times 415 \times 415 = 12309502009375$ the given *Resolvend*.

Example

Example 2. What's the *Sur-solid Root* of 2327834559873
The nearest *Sur-solid Number* to 232 is 243 whose *Root*
3 being *More than Just*:

$$\begin{array}{r} \text{Therefore} \quad 2430000000000 \\ - \quad 2327834559873 \\ \hline \text{Remains} \quad 102165440127 \text{ For a Dividend.} \end{array}$$

The *Cube* of 300 is 27000000 And $27000000 \times 5 = 135000000$
Then 135000000) 102165440127 (756,7810 New *Resolvend*.

$$\begin{array}{r} \text{First Root } 300) \quad 756,7810 \quad (2,558 \\ - 02 \times 2 = 4 \quad \underline{592} \\ \text{Divisor} \quad 296) \quad 164,78 \\ - ,5 \times 2 = 1,0 \quad \underline{147,50} \\ 2 \text{ Divisor} \quad 295,0) \quad 17,2810 \\ - ,05 \times 2 = ,10 \quad \underline{14,7450} \\ 294,90) \quad 2,53600 \text{ \&c.} \end{array}$$

The *First Root* was 300, being *More than Just*.

Therefore it is $- 02,558$

The *New Root* 297,442 And is very near the true
Root, which is 297,436 &c. Now the *Reason* why
this *Root* comes out to so many places of *Figures* at the *First*
Operation; is because the *First Sur-solid Number* was so near
the *Resolvend*, &c. As before.

Sect. 6. To **Extract** the *Root* of the **Square Cuber**,

This may be easily perform'd by *Two Extractions*; ac-
cording as its Name Denotes.

Thus, First *Extract* the *Square Root* of the given *Resol-*
vend; then *Extract* the *Cube Root* of that *Square Root*: And
it will be the *Root* required. That is, it will be the *Root* of
the *Sixth Power*,

Or thus, First *Extract* the *Cube Root* of the *Resolvend*,
then *Extract* the *Square Root* of that *Cube Root*: And it will
be the *Root* required.

Example 1. Let it be required to *Extract* the *Square Cubed*
Root out of this *Number* 145220337353515625 the *Resol-*
vend.

First I *Extract* the *Square Root*, of this *Resolvend*, which
take to be the best and the easiest way.

T

Thus

Thus

$$\begin{array}{r} 145220537353515625 \\ - 9 \\ \hline \text{Remains } 55220537353515625 \text{ To be halved:} \end{array}$$

$$\begin{array}{r} \text{Then } 3 \) \ 27610268676757812,5 \ (381078125 \\ + \underline{8} \quad \quad \quad 272 \\ \hline 38 \) \ 4102 \\ + \underline{10} \quad \quad \quad 3805 \\ \hline 3810 \) \ 2976867 \\ + \underline{7} \quad \quad \quad 2667245 \\ \hline 38107 \) \ 3096226 \\ + \underline{8} \quad \quad \quad 3048592 \\ \hline 381078 \) \ 47634757 \\ + \underline{1} \quad \quad \quad 38107805 \\ \hline 3810781 \) \ 95269528 \\ + \underline{2} \quad \quad \quad 76215622 \\ \hline 38107812 \) \ 1905390612,5 \\ + \underline{5} \quad \quad \quad 1905390612,5 \\ \hline 381078125 \quad \quad \quad (0) \end{array}$$

Having found the *Square Root* of the given *Resolvend*, proceed to *Extract* the *Cube Root* of that *Square Root*.

That is, of 381078125

$$- 343 = \text{the nearest Cube, its Root is } 700$$

$$\text{Then } 700 \times 3 = 2100 \) \ 38078125 \ (18161$$

$$\begin{array}{r} \text{First Root } 7 \) \ 18161 \ (25 \\ + \underline{2} \quad \quad \quad 144 \\ \hline 1. \text{ Divisor } 72 \) \ 3761 \quad \quad \text{First Root } 700 \\ + \underline{5} \quad \quad \quad 3625 \quad \quad \quad + \underline{25} \\ \hline 2. \text{ Divisor } 725 \quad \quad (136) \quad \quad \quad 725 \end{array}$$

Hence I find 725 to be the *Square Cube Root* required; as may easily be tried by *Involving* it to the *Sixth Power*.

That is $725 \times 725 \times 725 \times 725 \times 725 \times 725$ will be found $= 145220537353515625$ the given *Resolvend*.

Self. 7. To Extract the Root of the Seventh Power.

Having pointed the given *Resolvend*, as its *Index Denotes* *Viz.* into *Periods of Seven Figures*; Seek out such a *Number* of the *Seventh Power*; by the *Table of Powers*, as comes the nearest

nearest to the first Period of the Resolvend; whether it be Greater or Lesser, calling its respective Root More than Just, or Less than Just, annexing its proper Number of Cyphers, &c. As in the Cube and Sur-solid.

Then find the Difference between the given Resolvend, and that Number of the Seventh Power (found by the Table of Powers) by Subtracting the Lesser from the Greater.

Next find the Sur-solid or Fifth Power of that Root with its Annexed Cyphers (which you may also do by the Table of Powers) and Multiply that Sur-solid Number with 7 the Index of the given Resolvend, that Product must be a Divisor, by which the foresaid Difference must be Divided; that so it may be depressed to a Square, to be Pointed, &c. As before in the Cube, &c. Then make the First Root, without its Cyphers a Divisor Working with it and the New Resolvend, (as before,) only here you must Increase, or Diminish the Divisor with Thrice the Quotient Figure.

Example.

What's the Second Sur-solid Root, or that of the Seventh

Power, of 373236553955078125 the Resolvend pointed.

— 2178 the nearest Number of the Seventh Power.

155436553955078125 their Difference.

The First Root is 300 being Less than just; And the Fifth Power of 300 is 243000000000 which being Multiplied with 7 is 1701000000000 for a Divisor, by which the aforesaid Difference must be Divided; which Contracted may stand thus 1701) 15543655 (9137,95 &c.

First Root 3)	9137	(25	
+2 × 3 = 6	72		First Root = 300
1. Divisor 36	1937		+ 25
× 5 × 3 = 15	1875		True Root 325
2. Divisor 375	(62)		the Remainder to be rejected as before.

Hence I have found 325 to be the true Root required, that is, the true Root of the seventh Power.

I think it needless to proceed farther; viz. to Insert Examples of higher Powers. For if what is already done be well understood, it will be easy to conceive how to proceed in Extracting the Root of any Single Power how high soever it be (for the Method is General and alike in all Powers)

due regard being had to their *Indices*; and to the *first single side or Root*, That is, whether it be *More*, or *Less* than just &c.

Yet methinks I hear the young *Learner* say, 'tis possible to follow the *Directions* and *Examples*, as they are here laid down; but still here is not the Reason why they are so and so perform'd; And why there should be a *Remainder Left* after the *true Root* is found; viz. when the given *Resolvend* hath a *true Root* of its kind.

'Tis true, the *Reasons* of these are not here laid down neither indeed can they be rendered so plain and intelligible by words, as by an *Algebraick Process*, from whence the *Theorems* or *Rules* here given, had their *first Invention*; as shall be shewed in the next *Part*, when I come to treat of *Resolving Compounded or Adjected Equations*; however, take this *short* and *general* account of this *Method*.

This, and all other of the new *Methods* of *converging Series* (as they are called) are very different from the former (and still common) *Methods* of *Extracting Roots*, which requires the *first single side or Root* of the *first Period* (in any *Resolvend*) to be taken *Exactly true*, and then by *Involving* and other tedious ways of ordering it, there is formed a *Divisor*; which helps to grope out by *Trials* a *Second Figure* in the *Root*. And so proceeds on from *Point to Point*; still repeating the whole *Work* for every *Single Figure* that comes into the *Root*. And if there Chance to be a mistake or *Error* committed in any one *Figure* (as 'tis possible there may) it spoils the whole *Process*, which must then be wholly begun a new, or at least from that *Part* of it where the *Error* first entered.

But the *Nature* and *Design* of the *Method* which I have here laid down, is quite otherwise; it being so contrived as to gradually *Lessen* the *Difference* betwixt any proposed *Power*, and the like *Power* of another *Number* assumed; viz. it *Lessens* that *Difference* until it's either quite *vanquish'd*, or become so *Infinitely small* as to be *Insignificant*.

Therefore when any *Number* is proposed to have its *Root* *Extracted*; it is here required to take the next *nearest Root* of the *first Period* in the *Resolvend*; that so the *Difference* betwixt the given *Resolvend*, and the *Homogencal Power* (viz. the like *Power*) of the *Root* thus taken, may be *Left* either in *Excess*, or *Defect*. Which *Difference* being reduced or depressed *Lower*, becomes so prepared, that by plain *Division* (*comparatively*) there will arise such *Quotient Figures*

as will both *correct* and *Increase* the first Root to Three places of Figures at *Least*, sometimes to *Four*, or *Five* places of Figures; according as the said first *Difference* happens to be *More* or *Less*; (of which you may have observed Instances) But yet there will be a *Remainder* left, and perhaps an *Excess* or *Defect* in the Root so *increased*; viz. in the last Figure of it.

Now to *Rectifie* the said *Excess* or *Defect* in the Root, and to *discover* whether the given *Resolvend* be a true *Figurate Number* or not: That is, whether it have a true Root of it's kind. It will be *necessary* to make a *Second Operation*; by taking the Root so *increased*, and *proceeding* with it and the given *Resolvend*, in all respects as in the first Work (like to the *Third Example* of *Extracting the Cube Root*) I say, if the given *Resolvend* have a true Root, it will appear at this *Second Operation*, and all the *aforesaid Differences*, &c. will be *vanquished*; provided the Root required is not to have more than *Three* (or *Four*) places of Figures in it.

But if the Root be to have more than *Three Figures* in it; Or, that the given *Resolvend* prove to be a *Surd Number*. Then there will be a *Difference* as before; which will afford *Quotient Figures* to *Rectifie* and *Increase* the Root last taken, to three times as many places of Figures, as it had at the beginning of that *Second Operation*. As you may see in the *aforesaid Example 3.* of the *Cube Root*; wherein that Root is *Increased* to *Twelve* places of Figures at *Two Operations*; which if it were to be *Extracted* the old (and still *common*) way, it would require at least *forty times* the Number of Figures I have there used.

Again, if there *Chance* to be a *mistake* committed in any *Operation* perform'd by the *Method* here laid down, that *mistake* will not *destroy* the *precedent Work*, but will be *Rectified* in the next *Operation*, although it were not discovered before. And thus you may *proceed* on to a *Third Operation*, which will afford 27 places of Figures in the Root &c. with very little trouble if compared with former *Methods*.

This brief account, which I have here given; (by way of *Explaining the Nature of this Method of Extracting Roots*) being well *considered* and *compared* with the several *Operations* of the foregoing *Examples*: Must needs help the *Learner* to form such an *Idea* of it, that he cannot (*I presume*) but understand how to *proceed* in *Extracting* the Root out of any *Single Power*, how high soever it be; without the help of an *Algebraick Theorem*. Not, but when that comes
to

to be once understood ; the Work will be much readier and *easier perform'd* : As will appear in the *next part*.

I did intend to have here inserted, the whole *business* of *Interest* and *Annuities* ; But finding that it would require too *Large a Discourse*, to shew the *Grounds* and *Reasons* of the several *Theorems useful* therein. I have therefore reserved that Work for the close of the *next Part*. Neither indeed can the *Raising* of those *Theorems* be so well delivered in *words*, as by an *Algebraick* way of *Arguing* ; which renders them not only much *shorter*, but also *plainer* and *easier* to be understood.

I have also *Omitted* that *Rule* in *Arithmetick*, usually called the *Rule of Position* or *Rule of False*. Because all such *Questions*, as can be Answered by that *Guessing Rule* ; are much better done by any one who hath but a very small smattering of *Algebra*. I shall therefore conclude this part of *Numerical Arithmetick* ; And proceed to that of *Algebraick Arithmetick*, wherein I would advise the young *Learner* not to be too hasty in passing from one *Rule* to another, and then he will find it very easily to be attained.

A N

INTRODUCTION TO THE Mathematicks.

P A R T. II.

Proem.

Having formerly wrote a small Tract of Algebra perhaps it may seem (to some) very improper to write again upon the same Subject; but only (as the usual custom is) have refer'd my Reader to that Tract. However, because the following Parts of this Treatise, are managed by an Algebraick Method of Arguing; which may fall into the hands of those who have not seen that Tract; or any other of that kind: I thought it convenient to accommodate the young Geometer with the first Elements, or Principal Rules by which all Operations in this Art are perform'd: That so he may not be at a Loss as he proceeds further on; Besides, what I formerly wrote was only a Compendium of that which is here fully handled at large.

The Principal Rules are Addition, Substraction, Multiplication, Division, Involution, and Evolution; as in common Arithmetick (but differently perform'd) And therefore some call it Algebraick Arithmetick. Others call it Arithmetick in Species, because all the Quantities concern'd in any Question; Remain in their Substituted Letters (howsoever managed by Addition, Substraction, or Multiplication. &c.) without being destroy'd, or changed into others, as Figures in common Arithmetick are.

Mr. Harriot call'd it *Logistica Speciosa*, or *Specious Computation*.

C H A P.

C H A P. I.

Concerning the Method of Noting down Quantities ;
and Tracing their Steps. &c.

Sect. I. Of Notation.

The Method of Noting down Letters for Quantities, is various according to every ones fancy ; But I shall here follow the same as in my former Tract. And represent the Quantity sought (be it Line, or Number, &c) by the small (*a*) and if more Quantities than One are sought, represent them by the other small Vowels ; *e. u. or y.*

The given Quantities are represented by the small Consonants *b. c. d. f. g. &c.*

And for distinction's sake, mark the points or ends of Lines in all Schemes, with the Capital or great Letters, viz. *A. B. C. D. &c.*

When any Quantity (either given, or sought) is taken more than once, you must prefix its Number to it, As *3a* stands for *a* taken three times, or three times *a.* and *7b* stands for seven times *b* &c.

All Numbers thus prefixt to any Quantitie, are called Co-efficients, or Fellow-Factors ; because they Multiply the Quantity ; And if any Quantity be without a Co-efficient ; it is always suppos'd, or understood to have an Unit prefixed to it ; As *a* is *1a.* or *b* is *1b* &c.

The Signs by which Quantities are chiefly managed are the same ; and have the same Signification, with those in the first Part Page 5. which I here presume the Reader to be very well acquainted with, To them must be here Added these three more.

Viz. $\left\{ \begin{array}{c} \odot \\ \text{as} \\ \sqrt{} \end{array} \right\}$ the Sign $\left\{ \begin{array}{l} \text{Involution.} \\ \text{Evolution, or Extracting of Roots.} \\ \text{Irrationality, or Sign of a Surd Root.} \end{array} \right\}$ of

All Quantities that are express'd by Numbers only (as in Vulgar Arithmetick) are called Absolute Numbers.

Those Quantities that are represented by Single Letters as *a. b. c. d. &c.* or by several Letters that are immediately joyned together, As *ab. cd. or 7bd. &c.* are called Simple or Single whole Quantities.

But when different Quantities represented by different or unlike Letters, are connected together by the Sign (*+* or *-*) As *a + b. a - b. or ab - dc &c.* they are called Compound whole Quantities.

And when *Quantities* are *Express'd* or set down like *Vulgar Fractions*, Thus $\frac{a}{b}$. Or $\frac{a+b}{d}$. Or $\frac{ab+dc}{b-c}$. &c. they are called *Fractional* or *broken Quantities*.

The *Sign* wherewith *Quantities* are *Connected*, always belongs to that *Quantity* which immediately follows it ; And therefore all the *Quantities* concern'd in any *Question*, may stand in any order at *Pleasure*, viz. the most convenient for the *next Operation*. As $a+b-d$ may stand thus $b-d+a$ Or thus $a-d+b$ Or $-d+a+b$ &c. these being still the same tho' differently placed.

That *Quantity* which hath no *Sign* before it (as *Generally* the *Leading Quantity* hath not) is always understood to have the *Sign* $+$ before it, As a is $+a$ Or $b-d$ is $+b-d$ &c. for the *Sign* $+$ is the *Affirmative Sign*, and therefore all *Leading* or *Positive Quantities* are understood to have it, as well as those that are to be *Added*.

But the *Sign* $-$ being the *Negative Sign*, or *Sign of Defect*, there is a necessity of *prefixing* it before that *Quantity* to which it belongs, where-ever the *Quantity* stands.

SECT. 2. Of Tracing the Steps used in bringing Quantities to an Equation.

The *Method* of *Tracing* the *Steps*, used in bringing the *Quantities* concern'd in any *Question* to an *Equation*, is best perform'd by *Registering* the several *Operations*, with *Figures* and *Signs* placed in the *Margent* of the work, according as the several *Operations* require ; being very useful in long and tedious *Operations*.

For *Instance* : If it be required to set down, and *Register* the *Sum* of the *Two Quantities* a , and b , the *Work* will stand,

Thus	1	a
	2	b
$+2$	3	$a+b$

First set down the proposed *Quantities*, a and b over against the *Figures* 1 . 2 in the *Small Colum*, (which are here called *Steps*) and against 3 (the *Third Step*) Set down their *Sum*, viz. $a+b$. Then against that *Third Step*, Set down $1+2$ in the *Margin*; which Denotes that the *Quantities* against the *First* and *Second Steps* are added together, and that those in the *Third Step* are their *Sum*.

To *Illustrate* this in *Numbers*, Suppose $a=9$ and $b=6$ Then it will be,

Thus	1	a	$=9$
	2	b	$=6$
$+2$	3	$a+b$	$=9+6=15$

U

Again;

Again, If it were *required* to Set down the *Difference* of the same *Two Quantities*, Then it will be,

$$\begin{array}{r|l} \text{Thus} & \begin{array}{l} 1 \quad a = 9 \\ 2 \quad b = 6 \\ \hline 1 - 2 \quad 3 \quad a - b = 9 - 6 = 3 \end{array} \end{array}$$

Or if it were *required* to set down their *Product*. Then it will be,

$$\begin{array}{r|l} \text{Thus} & \begin{array}{l} 1 \quad a = 9 \\ 2 \quad b = 6 \\ \hline 1 \times 2 \quad 3 \quad a \times b \text{ or } ab = 9 \times 6 = 54 \end{array} \end{array}$$

&c.

Note, Letters set or joyn'd immediately together, (like a Word) signify the Rectangle or Product of those Quantities they Represent. As in the Last Example, wherein $ab = 54$ the Product of $a = 9$ and $b = 6$. &c.

Axioms.

1. If equal Quantities be Added to equal Quantities, the Sum of those Quantities will be equal.
2. If equal Quantities, be Taken from equal Quantities, the Quantities Remaining will be equal.
3. If equal Quantities, be Multiplied with equal Quantities, Their Products will be equal.
4. If equal Quantities, be Divided by equal Quantities, Their Quotients will be equal.
5. Those Quantities, that are equal to one and the same Thing, are equal to one another.

Note, I Advise the Learner to get these 5 Axioms perfectly by Heart.

These things being premised, and a perfect Knowledge of the Signs, and their Significations being gained, the Young Algebraist may proceed to the following Rules. But first I must make bold to Advise him here (as I have formerly done) that he be very Ready in one Rule, before he undertakes the next.

That is, He should be expert in Addition, before he meddles with Subtraction; And in Subtraction, before he undertakes Multiplication, &c. because they have a dependency one upon another.

C H A P. 2.

Concerning the Six Principal Rules, of Algebraick Arithmetick, of whole Quantities.

Se^t. i. Addition of whole Quantities.

Addition of whole Quantities admits of Three Cases.

Case 1. If the Quantities are like, and have like Signs; Add the Co-efficients or prefixt Numbers together; and to their Sum adjoyn the Quantities with the same Sign.

	Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
1	a	$- a$	$5b$	$- 7bc$
2	a	$- a$	$3b$	$- 8bc$
1 + 2	$2a$	$- 2a$	$8b$	$- 15bc$

	Exam. 5.	Exam. 6.	Exam. 7.
1	$3a + 5b$	$3a - 5b$	$6ab + 12$
2	$2a + 7b$	$2a - 7b$	$3ab + 24$
1 + 2	$5a + 12b$	$5a - 12b$	$9ab + 36$

The Reason of these Additions, is evident from the Work of Common Arithmetick. For suppose a , to represent one Crown, to which if I add One Crown, the Sum will be Two Crowns, or $2a$. As in Exam. 1.

Or if we suppose $-a$, to represent the want or debt of One Crown, to which if another want or debt of One Crown be added; the Sum must needs be the want or debt of Two Crowns, or $- 2a$. As in Exam. 2. And so for all the rest.

Case 2. If the Quantities are alike, And have unlike Signs; Subtract the Co-efficients from each other, and to their difference joyn the Quantities with the Sign of the Greater.

	Exam. 8.	Exam. 9.	Exam. 10.	Exam. 11.
1	$+ 5a$	$- 5a$	$7bc$	$- 9abd$
2	$- 3a$	$+ 3a$	$- 6bc$	$+ 7abd$
1 + 2	$- 2a$	$- 2a$	bc	$- 2abd$

	Exam. 12.	Exam. 13.
1	$7a - 5b$	$- 8ab - 7bc + 15$
2	$- 5a + 7b$	$+ 12ab + 7bc - 24$
1 + 2	$2a + 2b$	$4ab - 9$

The Reason of the Operations in this Case, may be easily understood by any one that duly considers the comparing of Stock, and Debts together; or the Ballancing of Accounts betwixt Debtor and Creditor.

That is, The Affirmative Quantities represent the Stock or Creditor; The Negative Quantities represent the Debts; And their Sum represents the Ballance, &c.

Case 3. When the Quantities are unlike, Set them all down, without altering their Signs; and thence will arise compound Quantities, which can be no otherwise Added but by their Signs.

Thus	1	a		a	$5b + 7dc$
	2	b		$-b$	$4a - 20$
1 + 2	3	$a + b$		$a - b$	$5b + 7dc + 4a - 20$

Here followeth a few Examples wherein all the 3 Cases are promiscuously concerned.

	1	$aa + 2ab + bb$		$8ab + bc - 37$
	2	$-4ab$		$-7ab - bc + 42 - 6d$
1 + 2	3	$aa - 2ab + bb$		$ab + 5 - 6d$

	1	$aa - 2ab + bb$		$9bc + 7ab - 45$
	2	$+4ab$		$4d - 6bc - 7ab + da$
1 + 2	3	$aa + 2ab + bb$		$3bc + 4d - 45 + da$

	1	$5a$		$a + b - ab$
	2	$-7a$		$7c - d$
	3	$+3a$		$4e + f$
1 + 2 + 3	4	a		$a + b - ab + 7c - d + 4e + f$

	1	$3aa + 4abc - bb + 30$
	2	$2bb - 3aa - 2abc - 25$
	3	$dd + 2aa - 3abc - 3$
1 + 2 + 3	4	$bb + dd + 2aa - abc + 2$

SECT. 2. Subtraction of whole Quantities.

Subtraction of whole Quantities, is performed by one General Rule.

Rule { Change all the Signs of the Subtrahend; (viz. of those Quantities which are to be Subtracted) or suppose them in your mind to be Changed. Then Add all the Quantities together, as before in Addition, and their Sum will be the true Remainder or Difference required.

This

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This General Rule is deduced from these evident truths.

To *Subtract* an *Affirmative Quantity*, from an *Affirmative*; is the same as to *Add* a *Negative Quantity* to an *Affirmative*.

That is $+2a$ Taken from $+3a$ is the same with $-2a$ Added to $+3a$.

Consequently, To *Subtract* a *Negative Quantity* from an *Affirmative*; will be the same as to *Add* an *Affirmative Quantity* to an *Affirmative*.

That is $-2a$ Taken from $+3a$ will be the same with $+2a$ Added to $+3a$.

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
	1	$2a$	$-2a$	$8b$	$-15bc$
	2	a	$-a$	$3b$	$-8bc$
-2	3	a	$-a$	$5b$	$-7bc$

		Exam. 5.	Exam. 6.	Exam. 7.
	1	$5a + 12b$	$5a - 12b$	$9ab + 36$
	2	$2a + 7b$	$2a - 7b$	$3ab + 24$
-2	3	$3a + 5b$	$3a - 5b$	$6ab + 12$

		Exam. 8.	Exam. 9.	Exam. 10.	Exam. 11.
	1	$+2a$	$-2a$	bc	$-2abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
-2	3	$+5a$	$-5a$	$+7bc$	$-9abd$

		Exam. 12.	Exam. 13.
	1	$2a + 2b$	$4ab - 9$
	2	$-5a + 7b$	$-8ab - 7bc + 15$
-2	3	$7a - 5b$	$12ab + 7bc - 24$

If these 13 Examples be Compared, with those in Addition; the Work will appear very evident, these being only the Converse or Proof of those; according to the Nature of Addition and Subtraction, in common Arithmetick.

More Examples in Subtraction.

	1	$a + b$	$5bc + 3da$	$8a + 5bd + 25$
	2	$a - b$	$5bc - 4da$	$7a - 3bd - 12$
-2	3	$-1 - 2b$	$+7da$	$a + 8bd + 37$

	1	$c + 13$	a	0
	2	$3a - b - 2c$	b	$2a - 4b$
1 - 2	3	$3c + 13 - 3a + b$	$a - b$	$-2a - 4b$

	1	$a + b - 54$		76
	2	$d - 3b - bc - 75$		$a - b - 5d - 7c$
1 - 2	3	$a + 4b + bc + 21 - d$		$76 - a + b + 5d - 7c$

That, $a - b$ Taken from $a + b$ Leaves $+2b$ for the Remainder; as in the first of these Examples, may be thus proved.

Let	1	$a + b = z$	
And	2	$a - b = x$	
2 + b	3	$a = x + b.$	per Axiom 1.
1 - 3	4	$b = z - x - b.$	per Axiom 2.
4 + b	5	$2b = z - x.$	which was to be proved.

The truth of all Operations in Subtraction, where any doubt arises; may be proved, by Adding the Subtrahend to the Remainder: As in common Arithmetick.

Examples.

	1	$+ 5a$	0	$- 9bc$	
	2	$- 2a$	$+ 3b$	$- 6da$	Subtrahend.
1 - 2	3	$+ 7a$	$- 3b$	$+ 6da - 9bc$	Remainder.
2 + 3	4	$+ 5a$	0	$- 9bc$	Proof.

SECT. 3 Multiplication of whole Quantities.

Multiplication of whole Quantities Admits of Three Cases.

Case 1. When the Quantities have like Signs, and no Co-efficients, set or joyn them together; And prefix the Sign + before them; and that will be their Product.

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
Thus {	1	a	$- a$	$a + b$	$- a - b$
	2	b	$- b$	d	$- d$
	1 x 2	3	ab	$- ab$	$ad + bd$

Case 2. If there be Co-efficients; Multiply them, and to their Product Adjoyn the Quantities set together as before.

The

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		Exam. 5.	Exam. 6.	Exam. 7.	Exam. 8.
Thus $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right.$	1	$5a$	$-6d$	$3a+2b$	$a+b$
	2	$3b$	$-7b$	6	$5b$
	3	$15ab$	$+42db$	$18a+12b$	$5ab+5bb$

Case 3. When the *Quantities* have *unlike Signs*; *Join* them and the *Product* of their *Co-efficients* together (as before) But *prefix* the *Sign* — before them;

		Exam. 9.	Exam. 10.	Exam. 11.	Exam. 12.
Thus $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right.$	1	$+a$	$-6d$	$4a-7b$	$4a-7b$
	2	$-b$	$-7b$	$3f$	$-3f$
	3	$-ab$	$-42db$	$12af-21bf$	$-12af-21bf$

That is, $+$ into $+$ or $-$ into $-$ gives $+$ } in the *Product*.
But $+$ into $-$ or $-$ into $+$ gives $-$

That $+$ into $+$ will *Produce* $+$ in the *Product*, is evident from *Multiplication* in *Common Arithmetick*. viz. $+5$ into $+7$ will give $+35$ &c.

But that $+$ into $-$ Or $-$ into $+$ should *Produce* the *Sign* — As in the *four last Examples*.

And that $-$ into $-$ should produce the *Sign* $+$ As in the *Second, Fourth and Sixth Examples*, may perhaps seem somewhat hard to be *Conceived*; and requires a *Demonstration*.

First, to prove that $-7b$ into $+3f = -21bf$. As in *Exam. 11*.

Suppose	1	$4a-7b=0$	
Then will	2	$4a=7b$	per Axiom 1.
But	3	$+3f=+3f$	
2×3	4	$12af=21bf$	per Axiom 3.
$4-21bf$	5	$12af-21bf=0$	per Axiom 2.

Consequently $+$ into $-$, Or $-$ into $+$ *Produces* $-$ which was the thing to be proved.

Secondly to prove $-7b$ into $-3f$ gives $+21bf$ as in *Exam. 12*.

Let	1	$4a-7b=0$	} as before
Then	2	$4a=7b$	
But	3	$-3f=-3f$	
the 2×3 is	4	$-12af=-21bf$	by what is proved above
$4+21bf$	5	$-12af+21bf=0$	per Axiom 1.

Consequently $-$ into $-$ gives $+$ which was to be proved.

Or

Or these may be otherwise proved by *Numbers*.

Thus, suppose $\begin{cases} a=20 \\ b=14 \end{cases}$ and $\begin{cases} c=12 \\ d=8 \end{cases}$ } or any other *Numbers*.

Then $a-b=6$ $c-d=4$ per *Axiom 2*.

Consequently, $a-b \times c-d = 6 \times 4 = 24$ per *Axiom 3*; but $a-b \times c-d$ according to the *Precedent Rules*, will be $ac - cb + bd - da$, which if true must be equal to 24.

$$\text{Proof } \begin{cases} ac = 20 \times 12 = 240 \\ bd = 14 \times 8 = 112 \end{cases} \quad \begin{cases} cb = 12 \times 14 = 168 \\ da = 8 \times 20 = 160 \end{cases}$$

Hence, $ac + bd = 352$ per *Axiom 1*.

And $cb + da = 328$ which being *Subtracted*

Leaves $ac + bd - cb - da = 352 - 328 = 24$ which plainly shews.

That $+$ into $-$ Produces $-$
And $-$ into $-$ Produces $+$ } in the *Product*.

Q. E. D.

Note. If the *Multiplier* consist of several *Terms*, then every one of those *Terms* must be *Multiplied* into all the *Terms* of the *Multiplicand*: And the *Sum* of those particular *Products*, will be the *Product* required. As in *Common Arithmetick*.

Examples.

	1	$a + b - d$	$7b + 5d$
	2	$a - b$	$3a - 5f$
$1 \times a$	3	$aa + ba - da$	$21ba + 15da$
$1 \times b$	4	$-ba - bb + db$	$-35bf - 25df$
$3 + 4$	5	$aa - da - bb - db$	$21ba + 15da - 35bf - 25df$

	1	$aa - ba$	$2c - 3d$
	2	$a + b$	$3a - 4b$
1×2	3	$aaa - abb$	$6ca - 9da - 8bc + 12db$

	1	$aa + 2a + 4$	$aa - ba + bb$
	2	$a - 2$	$a + b$
		$aaa + 2aa + 4a$	$aaa - baa + bba$
		$-2aa - 4a - 8$	$+baa - bba + bbb$
1×2	3	$aaa - 8$	$aaa + bbb$

Se^t. 4. Division of whole Quantities.

Division of Species, is the converse or direct contrary to that of Multiplication, and consequently perform'd by converse Operations. (As in common Arithmetick) And admits of Four Cases.

Case 1. When the Quantities in the Dividend, have like Signs to those in the Divisor, and no Co-efficients in either; Cast off or Expunge all the Quantities in the Dividend, that are like those in the Divisor; and set down the other Quantities with the Sign + for the Quotient required.

$$\text{Thus } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} ab \\ b \\ a \end{array} \left| \begin{array}{l} -ab \\ -b \\ +a \end{array} \right| \begin{array}{l} ad+bd \\ d \\ a+b \end{array} \left| \begin{array}{l} -ad-bd \\ -d \\ a+b \end{array} \right.$$

$$1 \div 2 \quad 3 \quad \begin{array}{l} a \\ +a \\ a+b \end{array} \quad \begin{array}{l} a+b \\ a+b \end{array}$$

Case 2. When the Quantities in the Dividend, have unlike Signs to those in the Divisor; Then set down the Quotient Quantities found as before, with the Sign - before them.

$$\text{Thus } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} +ab \\ -b \\ -a \end{array} \left| \begin{array}{l} -ab-bd \\ +b \\ -a-d \end{array} \right| \begin{array}{l} abc+bcd+bcf \\ -bc \\ -a-d-f \end{array}$$

$$1 \div 2 \quad 3 \quad \begin{array}{l} -a \\ -a-d \\ -a-d-f \end{array}$$

Case 3. If the Quantities in the Dividend and Divisor, have Co-efficients; Divide the Numbers (as in Common Arithmetick) and to their Quotient Adjoyn the Quotient Quantities.

$$\text{Thus } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} 15ab \\ 3b \\ 5a \end{array} \left| \begin{array}{l} 42db \\ -7b \\ -6d \end{array} \right| \begin{array}{l} 12af-21bf \\ 3f \\ 4a-7b \end{array}$$

$$1 \div 2 \quad 3 \quad \begin{array}{l} 5a \\ -6d \\ 4a-7b \end{array}$$

Note, When the Quantities and Co-efficients in the Divisor and Dividend are all the same, the Quotient will be in Unit or 1.

$$\text{Thus } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} ab \\ ab \\ 1 \end{array} \left| \begin{array}{l} 9bc \\ -9bc \\ -1 \end{array} \right| \begin{array}{l} 7ab+5bc \\ 7ab+5bc \\ 1 \end{array} \left| \begin{array}{l} 8ab+4d \\ -8ab-4d \\ -1 \end{array} \right.$$

$$1 \div 2 \quad 3 \quad \begin{array}{l} 1 \\ -1 \\ 1 \end{array} \quad \begin{array}{l} 1 \\ -1 \end{array}$$

Case 4. When the Quantities in the Divisor cannot be exactly found in the Dividend; then set them both down like a Vulgar Fraction. As in Common Arithmetick.

X

Thus

Thus {	1	a	$6bc$	$5b + 4a$	$8adc$
	2	b	$3d$	$5d + 7b$	$4abc$
$1 \div 2$	3	$\frac{a}{b}$	$\frac{2bc}{d}$	$\frac{5b + 4a}{5d + 7b}$	$\frac{2d}{b}$

N. B. In Division one thing must be very carefully observ'd; viz. that like Signs gives + and unlike Signs gives — in the Quotient; which needs no other Proof than that already laid down in the last Section, if duly compared with what hath been said concerning Multiplication and Division, in Vulgar Arithmetick.

Examples of Division at Large.

	1	$21ba + 15da - 35bf - 25df$	$(+ 3a$
	2	$7b + 5d$	
$2 \times 3a$	3	$21ba + 15da$	
$1 - 3$	4	$0 \quad 0 - 35bf - 25df$	$(- 5f$
$2 \times - 5f$	5	$- 35bf - 25df$	
$4 - 5$	6	$0 \quad 0$	$(5th Step.$
$1 \div 2$	7	$3a - 5f$	the Quotient collected from the 3d and

Or Division of Quantities may stand as Numbers in Common Arithmetick do; Thus

$$\begin{array}{r}
 3a - 6) 6aaaa - 96 \quad (2aaa + 4aa + 8a + 16 \\
 \underline{6aaaa - 12aaa} \\
 0 + 12aaa - 96 \\
 \underline{+ 12aaa - 24aa} \\
 0 + 24aa - 96 \\
 \underline{+ 24aa - 48a} \\
 0 + 48a - 96 \\
 \underline{+ 48a - 96} \\
 0 \quad 0
 \end{array}$$

That is, $6aaaa - 96 \div 3a - 6$ gives $2aaa + 4aa + 8a + 16$ for the Quotient, as may easily be proved by Multiplication. viz. $2aaa + 4aa + 8a + 16 \times 3a - 6$ will Produce $6a^4 - 96$ and so for the rest.

Sect. 5 Involution of whole Quantities.

Involution is the Raising or Producing of Powers, from any proposed Root, and is performed in all respects like Multiplication, save only in this; Multiplication Admits of any different Factors, but Involution still Retains the same.

Exam-

Examples.

	1	a	$-a$	the Root, or single Power.
1 ⊙ 2	2	aa	$+aa$	Square, or second Power.
1 ⊙ 3	3	aaa	$-aaa$	Cube, or third Power.
1 ⊙ 4	4	$aaaa$	$+aaaa$	Biquadrat, or 4th Power.
1 ⊙ 5	5	$aaaaa$	$-aaaaa$	Surfsolid, or 5th Power, &c.

Note, The Figures placed in the Margent, after the Sign (⊙) of Involution; shew to what height the Root is Involved; and are called Indices of the Power; and are usually placed over the Involved Quantities, in order to contract the Work, especially when the Powers are any thing high.

$$\text{Thus } \begin{cases} a = a \\ a^2 = aa \\ a^3 = aaa \\ a^4 = aaaa \end{cases} \quad \text{And } \begin{cases} a^5 = aaaaa \\ a^6 = aaaaaa \\ a^5 b^6 = aaaaaabbbbb \\ a^3 b^2 d^4 = aaabddddd \end{cases} \text{ \&c.}$$

If the Quantities have Co-efficients, they must be Involved along with the Quantities. As in these.

Thus	1	$2a$	$-3a$	$5bc$
1 ⊙ 2	2	$4aa$	$+9aa$	$25bbcc$
1 ⊙ 3	3	$8aaa$	$-27aaa$	$125bbbccc$
1 ⊙ 4	4	$16aaaa$	$+81aaaa$	$625b^4c^4$
1 ⊙ 5	5	$32aaaaa$	$-243a^5$	$3125b^5c^5$ &c.

Involution of Compound Quantities is performed in the same manner, due regard being had to their Signs and Co-efficients, if there be any.

As for instance, Suppose $a + b$ were given to be Involved to the 5th Power.

Thus	1	$a + b$	called a Binomial Root.
		$a + b$	
$\times a$	2	$aa + ab$	
$\times b$	3	$-ab + bb$	
⊙ 2	4	$aa + 2ab + bb$	the Square of $a + b$
		$a + b$	
$\times a$	5	$aaa + 2aab + abb$	
$\times b$	6	$+ aab + 2abb + bbb$	
⊙ 3	7	$aaa + 3aab + 3abb + bbb$	the Cube of $a + b$

X 2

aaa

	7	$aaa + 3aab + 3abb + bbb$ $a + b$
$7 \times a$	8	$a^4 + 3a^3b + 3aabb + abbb$
$7 \times b$	9	$+ a^3b + 3aabb + 3abbb + b^4$
$1 \ominus 4$	10	$a^4 + 4a^3b + 6aabb + 4abbb + b^4$ $a + b$
$10 \times a$	11	$a^5 + 4a^4b + 6a^3bb + 4aab^3 + ab^4$
$10 \times b$	12	$+ a^4b + 4a^3bb + 6aab^3 + 4ab^4 + b^5$
$1 \ominus 5$	13	$a^5 + 5a^4b + 10a^3bb + 10aab^3 + 5ab^4 + b^5$ $a + b \text{ \&c.}$

Again, Let $a - b$ called a *Residual Root*, be given

Then	1	$a - b$ $a - b$
$1 \times a$	2	$aa - ab$
$1 \times -b$	3	$- ab + bb$
$1 \ominus 2$	4	$aa - 2ab + bb$ the Square of $a - b$ $a - b$
$4 \times a$	5	$aaa - 2aab + abb$
$4 \times -b$	6	$- aab + 2abb - bbb$
$1 \ominus 3$	7	$aaa - 3aab + 3abb - bbb$ the Cube of $a - b$ $a - b$
$7 \times a$	8	$aaaa - 3aaab + 3aabb - abbb$
$7 \times -b$	9	$- aaab + 3aabb - 3abbb + bbbb$
$1 \ominus 4$	10	$aaaa - 4aaab + 6aabb - 4abbb + bbbb$ $a - b$
$10 \times a$	11	$a^5 - 4a^4b + 6a^3bb - 4aab^3 + ab^4$
$10 \times -b$	12	$- a^4b + 4a^3bb - 6aab^3 + 4ab^4 - b^5$
$1 \ominus 5$	13	$a^5 - 5a^4b + 10a^3bb - 10aab^3 + 5ab^4 - b^5$ $a - b \text{ \&c.}$

By comparing these Two Examples together, you may make the following Observations.

1. That the Powers raised from a *Residual Root* (viz. the Difference of Two Quantities) are the same with their Powers raised from a *Binomial Root* (or the Sum of Two Quantities) save only in their Signs; viz. the *Binomial Powers* have the Sign $+$ to every Term; But the *Residual Powers* have the Signs $+$ and $-$ interchangeably to every other Term.

2. The Indices of the Powers of the Leading Quantity (a) continually decrease in Arithmetical Progression; viz.

the Square it is aa, a In the Cube aaa, aa, a In the Biquadrat it is $aaaa, aaa, aa, a$ &c.

3. The *Indices* of the other Quantity b do continually increase in *Arithmetical Progression*; viz. In the Square it is b, bb In the Cube b, bb, bbb In the Biquadrat it is $b, bb, bbb, bbbb$ &c.

4. The first and last Terms, are always pure Powers of the single Quantities, and are both of the same height.

5. The Sum of the *Indices* of any two Letters joyned together in the intermediate Terms, are always equal to the Index of the highest Power, viz. of the first or last Term.

These Observations being duly considered, it will be easie to Conceive how the Terms of any proposed Power raised from a Binomial or Residual Root must stand, without their *Uncia* or Numeral Figures.

For instance, suppose it were required to Raise the Binomial Root $a + b$ to the Seventh Power; then the Terms of that Power will stand without their *Uncia*'s in this Order.

$$\text{Viz. } a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + ab^6 + b^7$$

And because the *Uncia* (not only of any Single Letter, but also) of every Single Power, how high soever it be, is an Unit or 1 (which neither Multiplies nor Divides) and all the Powers of any Binomial or Residual Root are naturally raised by Multiplying of the Precedent Power into it's Original Root, which is done by only joyning each Letter in the Root to the Precedent Power, with its *Uncia*, and then removing the said Power, when it is so joyn'd to the Second Letter, one place forwards (either to the Left, or Right Hand) it must needs follow.

That the *Uncia* of the Second Term (in any such Power) will always be the Sum of so many Units Added together more 1 as there hath been Multiplications of the first Root; which will always be determined by the Index of the first Term in the Power.

And because the *Unica*'s of all the intermediate Terms, are only removed along with their Letters, it also follows; that if they are Added together, their respective Sums will produce the true *Uncias* of the intermediate Terms in the new Raised Power. As doth plainly appear from the following Numbers so removed without their Letters; Which both shews and Demonstrates an easie way of producing the *Uncias* of any Ordinary Power (viz. of one not very high) Raised from either a Binomial, or Residual Root. Thus

Thus

Add $\begin{Bmatrix} 1.1. \\ 1.1 \end{Bmatrix}$ The two *Uncia's* of the first Root.

Add $\begin{Bmatrix} 1.2.1 \\ 1.2.1 \end{Bmatrix}$ The *Uncia's* of the Square.

Add $\begin{Bmatrix} 1.3.3.1 \\ 1.3.3.1 \end{Bmatrix}$ The *Uncia's* of the Cube.

Add $\begin{Bmatrix} 1.4.6.4.1 \\ 1.4.6.4.1 \end{Bmatrix}$ The *Uncia's* of the 4th Power.

Add $\begin{Bmatrix} 1.5.10.10.5.1 \\ 1.5.10.10.5.1 \end{Bmatrix}$ *Uncia's* of the 5th Power.

Add $\begin{Bmatrix} 1.6.15.20.15.6.1 \\ 1.6.15.20.15.6.1 \end{Bmatrix}$ *Uncia's* of the 6th Power.

$1.7.21.35.35.21.7.1$ *Uncia* of the 7th Power.

And so on in this manner *ad Infinitum*.

Now if these *Numbers* are prefixt to the aforesaid *Letters*, all the *Terms* will be compleated with their respective *Uncia's*, and will stand thus;

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

But that the business of finding these *Unicas*, may be rendered yet more easie for *Practice*, it will be convenient to consider what *Series* or *Progression*, the *Uncias* of each *Term* do make, from the aforesaid *Additions*.

<i>Uncia's</i> of the First Term.	<i>Uncia's</i> of the Second Term.	<i>Uncia's</i> of the Third Term.	<i>Uncia's</i> of the Fourth Term.	<i>Uncia's</i> of the Fifth Term.	<i>Uncia's</i> of the Sixth Term.	<i>Uncia's</i> of the Seventh Term.	<i>Uncia's</i> of the Eighth Term, &c.
1	1						<i>Uncia's</i> of the single Quantities.
1	2	1					<i>Uncia's</i> of the Square.
1	3	3	1				<i>Uncia's</i> of the Cube.
1	4	6	4	1			<i>Uncia's</i> of the 4th Power.
1	5	10	10	5	1		<i>Uncia's</i> of the 5th Power.
1	6	15	20	15	6	1	<i>Uncia's</i> of the 6th Power.
1	7	21	35	35	21	7	<i>Uncia's</i> of the 7th Power. &c.

The *Uncias* of the first Term is only a *Series* of *Units*, whose *Sum* is every where the *Uncias* of the Second Term.

The *Uncias* of the Second Term, is a *Series* of *Numbers* in *Arithmetick Progression*; whose *Sum* is every where the *Uncia* of the next Superiour Power in the third Term, and may be found by *Proposition* 1. Chap. 6. Part 1.

That is, in the 7th Power it will be $\frac{6+1 \times 6}{2} = 21$
the *Uncia* of the Third Term.

The rest of the *Uncias* are a Compounded Series, whose
respective Sums may be obtained from the *Uncia's* of their
precedent Terms.

Thus $\frac{21 \times 5}{3} = 35$ Then $\frac{35 \times 4}{4} = 35$ Again $\frac{35 \times 3}{5} = 21$

And $\frac{21 \times 2}{6} = 7$ &c.

From hence may be deduced this General Rule.

Rule. { If the Index of the first Letter of any Term, be
Multiplied into its own *Uncia*, and that Product
be Divided by the Number of Terms to that place;
the Quotient will be the *Uncia* of the next suc-
ceeding Term forward.

That is, by the help of those *Indices* that belong to the
several Powers of the first or Leading Letter only (as *a*)
the true *Uncia's* of every Term may be easily found.

Example.

Let it be required to compleat all the Terms of the afore-
said several Powers; viz. $a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4$
 $+ a^2b^5 + ab^6 + b^7$ with their proper *Uncia's*.

1. The Index of a^7 the First Term will be the *Uncia*
of the Second Term. Thus $a^7 + 7a^6b$.

2. Then half the Second Terms Index into its *Uncia*,
viz. $\frac{7 \times 6}{2} = 21$ will be the Third Terms *Uncia*.

Thus $a^7 + 7a^6b + 21a^5b^2$ will be the three first Terms.

3. Again $\frac{21 \times 5}{3} = 35$ is the *Uncia* of the fourth Term.

Then it will be $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3$

4. And $\frac{35 \times 4}{4} = 35$ will be the *Uncia* of the fifth Term.

Then $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4$ &c. un-
till all the Terms are compleated with their respective
Uncias; and then they will stand

Thus $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 +$
 $21a^2b^5 + 7ab^6 + b^7$.

Now

Now here it may be further observed, that the *Uncias* do only increase until the *Indices* of the two *Letters* become equal, or change places; And then the rest of the *Uncias* will return or decrease in the same order. That is, wherever the *Indices* of the *Letters* are alike, there the *Uncias* will be alike.

And therefore one needs to find the *Uncias* (as before) but to half the *Number* of *Terms* in any *Power*.

If what hath been here said, and the *Work* of the *Example* be well understood, I presume it will be found very easie to Raise any *Power* from a *Binomial* or *Residual Root*, to what height you please; without the trouble of a continued *Involution*; and without the help of such a *Table* of *Powers* as is proposed by Mr. Oughtred in his *Key to the Mathematicks*, Page 40. and since by others.

Now from these *Considerations* it was, that I proposed this *Method* of raising *Powers* in my *Compendium of Algebra* Page 47, as wholly new (viz. so much of it as was there useful) having then (I profess) neither seen the way of doing it, nor so much as heard of its being done. But since the writing of that *Tract*. I find in Dr. Wallis's *History of Algebra*; Page 319 and 331 that the *Learned* Mr. Isaac Newton had discovered it long before: Which the *Doctor* sets down in this manner.

Let m be the *Exponent* of the *Power*.

$$\text{Then } \left\{ 1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \text{ \&c.} \right.$$

Will be the *Series* of the *Uncias* required; But he doth not tell us how they came to be first found out, nor have I ever met with the least hint of it in any *Author*.

Sect. 6. Evolution of whole Quantities.

Evolution is the *Extracting* of *Roots* from any given *Power*. That is, it is the *Converse Work* to that of *Involution* and in *Single Quantities* it's easie, if the given *Power* have such a *Root* as is required, which may be thus known.

If the given *Power* have no *Numbers* prefix'd to it, and its *Index* can be *Divided* by the *Index* of the *Root* required, the *Quotient* will be the *Index* of the *Root* sought.

Thus, If the *Cube Root* of *aaaaaa* viz. a^6 were required (the *Index* of the *Cube* is 3) then 3) 6 (2 That is 3) a^6 (a^2 the *Root* required. And such *Operations* are usually set down.

Thus	1	a^6	a^6b^6	$a^6b^6d^6$
1 w 2	2	a^3	a^3b^3	$a^3b^3d^3$
1 w 3	3	aa	$aabb$	$aabbdd$
3 w 2	4	a	ab	abd

Note, the Figures placed in the Margin after the Sign (*w*) of Evolution, denote the Index of the Root to be Extracted.

If the given Powers have Co-efficients; (viz. Numbers prefix'd to them;) then you must Extract their respective Roots as in *Vulgar Arithmetick*.

Thus	1	$81a^4$	$1296a^8b^8$	$20736a^4b^4c^4$
1 w 2	2	$9aa$	$36a^4b^4$	$144aabbcc$
1 w 4	3	$3a$	$6aabb$	$12abc$
Or 2 w 2	4	$3a$	$6aabb$	$12abc$

But if the Root required cannot be truly Extracted out of both the Co-efficients and Indices of the given Power; Then it is a *Surd*, and must have the Sign of the Root required prefix'd to it.

Thus	1	a^5	$67aaaa$	$216bbbddd$
1 w 2	2	$\sqrt{a^5}$	$\sqrt{67aaaa}$	$\sqrt{216bbbddd}$
1 w 3	3	$\sqrt[3]{a^5}$	$\sqrt[3]{67aaaa}$	$6bd$

Evolution of Compound Quantities or Powers, is a little more troublesome than that of Single Powers; and would require a great many words to Explain the manner, and Reason of forming the several Cannons, that are commonly used in Extracting the Roots of Compound Quantities; especially if the Powers be very high, &c. I shall therefore for brevities sake omit them, and instead thereof, propose an easie Method of Discovering the Roots of all Compound Powers in general; And in order to that, it will be necessary to premise; That if either the Sum, or difference of several Quantities be Involved to any Power, there will arise so many Single Powers of the same height, as there are different Quantities.

As for instance, if $a+b+d$ be squared; That is, be Involved to the Second Power, it will be $aa+2ab+2ad+bb+2bd+dd$: here you have aa , bb , and dd .

Again, if $a+b+d$ were cubed, viz. Involved to the Third Power, then you will have aaa , bbb , and ddd , in it, &c.

Y

Whence

Whence it follows, that in *Extracting* the *Roots* of all *Compound Quantities*, there must be consider'd.

1. How many *different Letters* (or *Quantities*) there are in the given *Power*.

2. Whether the *Single Powers* of each of those *Letters* be of an *equal height* and have in them such a *single Root* as is required: Which if they have, *Extract* it as before.

3. Connect those *Single Roots* together with the *Sign* + and *Involve* them to the same *height* with the given *Power*; that being done, compare the new *Raised Power* with the given *Power*, and if they are alike in all their respective *Terms*, then you have the *Root* required; Or if they differ only in their *Signs*, the *Root* may be easily corrected with the *Sign* — as occasion requires.

Example 1. Let it be required to *Extract* the *Square Root* of $cc + 2cb - 2cd - bb - 2bd + dd$.

In this *Compound Square* there are *Three Distinct Powers*; viz. bb , cc , dd , whose *Single Roots* are b , c , d , wherefore I suppose the *Root* sought to be $b + c + d$, or rather $b + c - d$, because in the given *Power* there is $-2cd$, and $-2bd$, therefore I conclude it is $-d$, then $b + c - d$ being squared, Produces $bb + 2bc - 2bd + cc - 2cd + dd$ which I find to be the same in all its *Terms*, with the given *Power*, although they stand in a *different Position*; consequently $b + c - d$ is the true *Root* required.

Example 2. 'Tis required to *Extract* the *Square Root* of $a^4 - 2aabb + b^4$. Here are but two *Single Powers*, viz. a^4 , and b^4 , whose *Square Roots* are aa , and bb . And because in the given *Power* there is $-2aabb$, therefore I conclude it must either be $aa - bb$, Or $bb - aa$. Both which being *Involved* will Produce $a^4 - 2aabb + b^4$. Consequently the *Root* sought may either be $aa - bb$ Or $bb - aa$ according to the *nature* or *design* of the *Question*, from whence the given *Power* was produced.

Example 3. Let it be required to *Extract* the *Square Root* of $36aaaa + 108aa + 81$. Here the *Two Single Powers* are $36aaaa$. And 81 , whose *Roots* are $6aa$ and 9 . And because the *Signes* are all + therefore I suppose the *Root* to be $6aa + 9$ the which being *Involved* doth Produce $36a^4 + 108aa + 81$, Consequently $6aa + 9$ is the true *Root* required.

Example 4. Suppose it were required to *Extract* the *Cube Root* of $125aaa + 300aae - 450aa + 240aee - 720ae + 64eee + 540a - 288ee + 432e - 216$.

In this Example there is Three Distinct Powers, viz. $125aaa$, $64eee$, and -216 .

The Cube Root of $125aaa$ is $5a$, Of $64eee$ is $4e$ and the Cube Root of -216 is -6

Wherefore I suppose the Root sought to be $5a + 4e - 6$ which being Involved to the Third Power, does produce the same with the given Power, consequently $5a + 4e - 6$ is the Cube Root required.

But if the new Power, raised from the supposed Root (being Involved to its due height) should not prove the same with the given Power, viz. if it hath either More, or Fewer Terms in it, &c. Then you may conclude the given Power to be a Surd, and must have its proper Sign prefix'd to it, and cannot be otherwise Express'd; until it come to be Evolved in Numbers.

Example 5. Suppose it were required to Extract the Cube Root of $27aaa + 54baa + bbb$.

Here is Two Distinct and perfect Cubes, viz. $27aaa$, and $8bbb$ whose Cube Roots are $3a$ and $2b$.

Wherefore one may suppose the Root sought to be $3a + 2b$ which being Involved to the Third Power, is $27aaa + 54baa + 36bba + 8bbb$. Now this new raised Power hath one Term (viz. $36bba$) more in it than the given Power hath; but this being a perfect Cube, one may therefore conclude the given Power is not so, viz. it is a Surd, and hath not such a Root as was required, but must be express'd or set down.

Thus $\sqrt[3]{27aaa + 54baa + 8bbb}$

If these Examples be well understood, the Learner will find it very easie, by this Method of proceeding to discover the true Root of any given Power whatsoever.

C H A P. III.

Of Algebraick Fractions, or Broken Quantities.

SECT. I. Notation of Fractional Quantities.

Fractional Quantities are express'd or set down like Vulgar Fractions in common Arithmetick.

Thus $\left\{ \begin{array}{l} \frac{a}{b} \quad \frac{2bc}{d} \quad \frac{5b-4a}{4d+7b} \end{array} \right.$ Numerators.
Denominators.

Y 2

How

How they come to be so, see *Case 4.* in the Last Chapter of *Division*. These *Fractional Quantities* are managed in all respects like *Vulgar Fractions* in *Common Arithmetick*.

Sect. 2. To Alter or Change different *Fractions* into one *Denomination*, retaining the same value.

Rule. $\left\{ \begin{array}{l} \text{Multiply all the Denominators into each other for a} \\ \text{new Denominator; And each Numerator into all} \\ \text{the Denominators but its own, for new Numerators.} \end{array} \right.$

Examples.

Let it be required to bring $\frac{a}{b}$ and $\frac{d}{c}$ into one *Denomination*.

First $a \times c$, and $d \times b$, will be the *Numerators*, And $b \times c$ will be the *common Denominator*, viz. $\frac{ca}{bc}$ and $\frac{bd}{bc}$ are the

Two Fractions required. That is $\frac{ca}{bc} = \frac{a}{b}$ and $\frac{bd}{bc} = \frac{d}{c}$

Again, let $\frac{b+c}{a+b}$ and $\frac{d-c}{b-d}$ be brought into one *Denomination*.

And they will be $\left\{ \begin{array}{l} \frac{bb+bc-bd-dc}{ba+bb-da-bd} \text{ and } \frac{ad-ac+bd-bc}{ba+bb-ad-bd} \end{array} \right.$ &c.

Sect. 3. To Bring whole *Quantities* into *Fractions* of a given *Denomination*.

Rule. $\left\{ \begin{array}{l} \text{Multiply the whole Quantities into the given Denominator, for a Numerator; under which subscribe} \\ \text{the given Denominator, and you will have the Fraction required.} \end{array} \right.$

Examples.

Let it be required to bring $a+b$ into a *Fraction* whose *Denominator* is $d-a$ First $a+b \times d-a$ is $da+bd-aa-ba$

Then $\frac{da+bd-aa-ba}{d-a}$ is the *Fraction* required.

Again $b + \frac{a}{d}$ will be $\frac{db+a}{d}$. And $\frac{aa}{d} - a$ will be $\frac{aa-da}{d}$. Also $a+b + \frac{aa+bb}{a-b}$ will be $\frac{2aa}{a-b}$

When

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When whole Quantities are to be set down Fraction-wise, Subscribe an Unit for the Denominator

Thus ab is $\frac{ab}{1}$ And $aa - bb$, is $\frac{aa - bb}{1}$ &c.

Sect. 4. To Abbreviate or Reduce Fractional Quantities into their Lowest Denomination.

Rule. $\left\{ \begin{array}{l} \text{Divide both the Numerator and Denominator by} \\ \text{their greatest common Divisor; viz. by such Quan-} \\ \text{tities as are found in both; and their Quotients} \\ \text{will be the Fraction in its Lowest Term,} \end{array} \right.$

Thus $\frac{aac}{dc}$ is $\frac{aa}{d}$ And $\frac{abbb}{abc}$ is $\frac{bb}{c}$ Also $a + \frac{bdc}{bc} = a + d$

In such Single Fractions as these; the common Divisors (if there be any) are easily discover'd by inspection only; but in Compound Fractions it often proves very troublesome, and must be done either by Dividing the Numerator by the Denominator until nothing Remains, when that can be done: Or else finding their common Measure by Dividing the Denominator by the Numerator, and the Numerator by the Remainder, and so on as in Vulgar Fractions (Sect. 4. Page 51.)

Examples.

Suppose $\frac{aac - aad}{cd - dd}$ were to be Reduced Lower.

Then $cd - dd$) $\frac{aac - aad}{aac - aad}$ $\left(\frac{aa}{d} \right.$ the Fraction required,

In this Example it so happens that the Numerator is Divided just off by the Denominator; but in the next its other wise, and requires a double Division to find out the common Measure. Viz. Let it be required to Reduce $\frac{aaa - abb}{aa + 2ab + bb}$ to its Lowest Terms.

First $aa + 2ab + bb$) $\frac{aaa - abb}{aaa + 2aab - abb}$ $\left(a \right.$
 $\quad \quad \quad - 2aab - 2abb$ the Remainder.
 Then $- 2aab - 2abb$) $\frac{aa + 2ab + bb}{aa + ab}$ $\left(\frac{1}{2b} - \frac{1}{2a} \right.$
 $\quad \quad \quad ab + bb$
 $\quad \quad \quad ab - bb$
 $\quad \quad \quad 0 \quad 0$

Hence

Hence it appears that $-2aab - 2abb$ is the common Measure; by which $aaa - abb$ being Divided.

$$\begin{array}{r} \text{Viz. } -2aab - 2abb \) \ aaa - abb \quad \left(-\frac{a}{2b} + \frac{1}{2} \right. \\ \underline{aaa + aab} \\ -aab - abb \\ \underline{-aab - abb} \\ 0 \end{array}$$

Then $-\frac{a}{2b} + \frac{1}{2}$ is the New Numerator.

And $-\frac{1}{2b} - \frac{1}{2a}$ is the New Denominator.

But $-\frac{a}{2b} + \frac{1}{2} = -\frac{2a + 2b}{4b} = -\frac{a + b}{2b}$ the Numerator

And $-\frac{1}{2b} - \frac{1}{2a} = -\frac{2a - 2b}{4ba} = -\frac{a - b}{2ba}$ the Denominator.

Let both be Multiplied with $2ba$ and you will have
 $-aa + ab$ the Numerator.
 $-a - b$ the Denominator.

Or Changing the Signs of all the Quantities, and it will be $\frac{aa - ab}{a + b}$ the New Fraction required.

$$\text{That is } \frac{aa - ab}{a + b} = \frac{aaa - abb}{aa + 2ab + bb}$$

Again let it be required to Reduce $\frac{dd - bb}{ddd - bbb}$ The common Measure of this Fraction will be the easiest found (as appears from Trials) by Dividing the Denominator by the Numerator, &c.

$$\begin{array}{r} \text{Thus } dd - bb \) \ ddd - bbb \quad (d \\ \underline{ddd - bbd} \\ + bbd - bbb \\ \underline{+ bbd - bbb} \\ 0 \end{array} \quad \begin{array}{r} dd - bb \quad \left(\frac{d}{bb} \right. \\ \underline{dd - bd} \\ + bd - bb \\ \underline{+ bd - bb} \\ bbd - b^3 \quad (b \\ \underline{bbd - b^3} \\ 0 \end{array}$$

Hence it appears that $bd - bb$ is the common Measure that will Divide both the Numerator and the Denominator.

Consequently

Examples in Subtraction.

1 — 2	1	$\frac{bb + aa}{c}$	$\frac{a + b}{d + c}$	$\frac{3a + b + c}{d}$	$\frac{2a}{d + a}$
	2	$\frac{bb}{c}$	$\frac{2b - a}{d + c}$	$\frac{2a + c}{d}$	$\frac{a + b - d}{d + a}$
	3	$\frac{aa}{c}$	$\frac{2a - b}{d + c}$	$\frac{a + b}{d}$	$\frac{a - b + d}{d + a}$

Sect. 6. Multiplication of Fractional Quantities.

First prepare *Mix'd Quantities* (if there be any) by making them *improper Fractions*, and whole *Quantities* by subscribing an *Unit* under them. As per Sect. 3. Then

Rule. $\left\{ \begin{array}{l} \text{Multiply the Numerators together for a new Nu-} \\ \text{merator: And the Denominators together for a} \\ \text{New Denominator. As in Vulgar Fractions.} \end{array} \right.$

Thus

1 × 2	1	$\frac{ab}{c}$	$\frac{3a - 2b}{2d + c}$
	2	$\frac{d}{f}$	$\frac{4a + 2b}{d}$
	3	$\frac{abd}{cf}$	$\frac{12aa - 2ab - 4bb}{2dd + dc}$

Suppose it were required to Multiply $2a + \frac{b}{c} - 25$ with $3b + 4c$. These prepared for the Work (per Sect. 3) will stand.

Thus

1 × 2	1	$\frac{2ac + b - 25c}{c}$
	2	$\frac{3b + 4c}{1}$
	3	$\frac{6bac + 3bb - 75bc + 81cc + 4bc - 100cc}{c}$
Or 3	4	$6ba - 71b + 8ac - 100c + \frac{3bb}{c}$ per Sect. 4

N. B. Any Fraction is Multiplied with its Denominator by casting off, or Taking the Denominator away.

Thus $\frac{b}{a} \times a$ gives b . For $\frac{b}{a} \times \frac{a}{1} = \frac{ba}{a} = b$ &c.

Sect. 7. Division of Fractional Quantities.

The Fractional Quantities being prepar'd, as directed in the Last Section. Then

Rule. $\left\{ \begin{array}{l} \text{Multiply the Numerator of the Dividend, into} \\ \text{the Denominator of the Divisor; for a New} \\ \text{Numerator; And Multiply the other two toge-} \\ \text{ther for a New Denominator. As in Vulgar} \\ \text{Fractions.} \end{array} \right.$

Examples.

Let $\frac{abd}{cf}$ be Divided by $\frac{ab}{c}$ the work may stand

Thus $\frac{ab}{c} \bigg) \frac{abd}{cf} \left(\frac{abdc}{abcf} = \frac{d}{f}$ per Sect. 4.

Or Thus	1	$\frac{abd}{cf}$	$\frac{a+b}{d}$	$\frac{aaa-bbb}{a+b}$
	2	$\frac{ab}{c}$	$\frac{c-b}{a}$	$\frac{aa-ab+bb}{c}$
	3	$\frac{d}{f}$	$\frac{aa+ba}{dc-ab}$	$\frac{aaac-bbbc}{aaa+bbb}$

Suppose it were required to Divide $aa + \frac{3abb}{a+4b}$

By $a+b$. Then the Work prepared will stand

Thus $\frac{a+b}{1} \bigg) \frac{aaa+4aab+3abb}{a+4b} \left(\frac{aaa+4aab+3abb}{aa+5ba+4bb}$

But $\frac{aaa+4aab+3abb}{aa+5ba+4bb} = \frac{aa+3b}{a+4b}$ (per Sect. 4.)

When Fractions are of one Denomination, cast off the Denominators, and Divide the Numerators.

Thus, if $\frac{ab^3}{c}$ were to be Divided By $\frac{bb}{c}$ it will be

$bb) ab^3$ ($\frac{ab}{2}$ the Quotient required. For

For $\frac{bb}{c} \div \frac{ab^3}{c} \left(\frac{ab^3c}{bbc} \right)$ But $\frac{ab^3c}{bbc} = ab$ (per Sect. 4.)

Again, Suppose it were required to Divide $\frac{a^3 - abb}{c - d}$

By $\frac{aa + 2ab + bb}{c - d}$ casting off $c - d$ in both, it will be

$$aa + 2ab + bb \bigg) aaa - abb \quad \left(\frac{aa - ba}{a + b} \text{ \&c.} \right)$$

Sect. 8. Involution of Fractional Quantities.

Rule. $\left\{ \begin{array}{l} \text{Involve the Numerator into it self, for a new Nu-} \\ \text{merator; and the Denominator into it self, for a new} \\ \text{Denominator; each so often as the Power requires.} \end{array} \right.$

Thus	1	$\frac{b}{a}$	$\frac{3bc}{2ad}$	$\frac{b+d}{a-c}$
1 $\odot$ 2	2	$\frac{bb}{aa}$	$\frac{9bbcc}{4aadd}$	$\frac{bb+2bd+dd}{aa-2ac+cc}$
1 $\odot$ 3	3	$\frac{bbb}{aaa}$	$\frac{27bbbccc}{8aaaddd}$	$\frac{bbb+3bbd+3bdd+ddd}{aaa-3aac+3acc-ccc}$

Sect. 9. Evolution of Fractional Quantities:

If the Numerator and Denominator of the given Fraction, have each of them such a Root as is required; (which very rarely happens.) Then Evolve them; and their respective Roots will be the Numerator, and Denominator of the new Fraction required.

Thus	1	$\frac{9aabb}{4dd}$	$\frac{aa+2ab+bb}{aa-2ab+bb}$
1 ω 2	2	$\frac{3ab}{2d}$	$\frac{a+b}{a-b}$

Again	1	$\frac{27aabb^3}{8ddd}$	$\frac{aaa+3aab+3abb+bbb}{aaa-3aab+3abb-bbb}$
1 ω 3	2	$\frac{3ab}{2d}$	$\frac{a+b}{a-b}$

Sometimes it so falls out, that the Numerator may have such a Root as is required, when the Denominator hath not; Or

Or the *Denominator* may have such a *Root*, when the *Numerator* hath not. In those *Cases* the *Operations* may be set down.

Thus	1	$\frac{aabb}{ddd}$	$\frac{aaa + 4bb - dd}{aa + 2ab + bb}$
I w 2	2	$\frac{ab}{\sqrt{ddd}}$	$\frac{\sqrt{aaa + 4bb - dd}}{a + b}$

But when neither the *Numerator*, nor the *Denominator* have just such a *Root* as is required; prefix the *Radical Sign* of the *Root* to the *Fraction*; and it then becomes a *Surd*, as in the last Step, which brings me to the business of managing *Surds*.

CHAP. IV.

Of Surd Quantities.

The whole *Doctrin* of *Surds* (as they call it) were it fully handled, would require a very *Large Explanation* (to render it but *tollerably intelligible*) even enough to fill a *Treatise* it self; if all the *various Examples* that may be of use to make it *Easie* should be inserted; without which it's very *intricate* and *troublesom* for a *Learner* to understand. But now those *tedious Reductions* of *Surds*, which were heretofore though *useful* to fit *Equations* for such a *Solution*, as was then *understood*, are wholly laid aside as *useless*: Since the *New Methods* of *Resolving* all sorts of *Equations* renders their *Solutions* equally *easie*, altho' their *Powers* are never so *high*.

Nay, even since the true use of *Decimal Arithmetick* hath been well *understood*, the business of *Surd Numbers* has been managed that way; as appears by several instances of that kind, in *Doctor Wallis's History of Algebra*, from page 23 to 29.

I shall therefore for *brevities sake*, pass over those *tedious Reductions*, and only shew the *Young Algebraist* how to deal with such *Surd Quantities* as may arise in the *Solution* of hard *Questions*.

SECT. I. Addition and Substraction of Surd Quantities.

Case 1. When the *Surd Quantities* are *Homogenial* (*viz.* alike) Add or Substract the *Rational Part*, if they are joyned

joyned to any, and to their *Sum*, or *Difference*, *Adjoin* the *Irrational* or *Surd*.

Examples in Addition.

I + 2	1	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
	3	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$

I + 2	1	$4d\sqrt[3]{aa}$	$b + \sqrt[3]{aa - cc}$	$bc\sqrt[3]{aa + d}$
	2	$d\sqrt[3]{aa}$	$c - \sqrt[3]{aa - cc}$	$3bc\sqrt[3]{aa + d}$
	3	$5d\sqrt[3]{aa}$	$b + c$	$4bc\sqrt[3]{aa + d}$

Examples in Subtraction.

I - 2	1	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
	3	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$

I - 2	1	$5d\sqrt[3]{aa}$	$b + c$	$4bc\sqrt[3]{aa + d}$
	2	$4d\sqrt[3]{aa}$	$c - \sqrt[3]{aa - cc}$	$3bc\sqrt[3]{aa + d}$
	3	$d\sqrt[3]{aa}$	$b + \sqrt[3]{aa - cc}$	$bc\sqrt[3]{aa + d}$

Case 2. When the *Surd Quantities* are *Herterogenia* (viz. their *Indices* are *unlike*) they are only to be *Added* or *Subtracted* by their *Signs*, viz. $+$ Or $-$ And from thence will arise *Surds* either *Binomial*, or *Residual*.

Examples in Addition.

I + 2	1	$\sqrt{bc}$	$4d\sqrt{a}$	$\sqrt{ac - ba}$
	2	$\sqrt{ba}$	$2b\sqrt{ac}$	$\sqrt{ac + ba}$
	3	$\sqrt{bc} + \sqrt{ba}$	$4d\sqrt{a} + 2b\sqrt{ac}$	$\sqrt{ac - ba} + \sqrt{ac + ba}$

Examples in Subtraction.

I - 2	1	$\sqrt{bc}$	$b - d\sqrt{aaa + ca}$
	2	$\sqrt{ba}$	$d - 2a\sqrt{bd + dd}$
	3	$\sqrt{bc} - \sqrt{ba}$	$b - d\sqrt{aaa + ca} - d + 2a\sqrt{bd + dd}$

Sect. 2. Multiplication of Surd Quantities.

Case 1. When the Quantities are Pure Surds of the same kind; Multiply them together, and to their Product prefix their Radical Sign.

Examples.

1 × 2	1	$\sqrt{b}$	$\sqrt{ba + da}$	$\sqrt{aa + bb}$
	2	$\sqrt{a}$	$\sqrt{ca}$	$\sqrt{aa - bb}$
	3	$\sqrt{ba}$	$\sqrt{bcaa + dcaa}$	$\sqrt{aaaa - bbbb}$

Case 2. If Surd Quantities of the same kind (as before) are joyned to Rational Quantities; then Multiply the Rational into the Rational; and the Surd into the Surd, and joyn their Products together

Examples.

1 × 2	1	$d\sqrt{bc}$	$5cd\sqrt{ba + da}$	$15\sqrt{ab}$
	2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
	3	$3db\sqrt{bca}$	$15cda\sqrt{bcaa + dcaa}$	$75\sqrt{abd}$

Sect. 3. Division of Surd Quantities.

Case 1. When the Quantities are Pure Surds of the same kind, and can be Divided off (viz. without leaving a Remainder) Divide them, and to their Quotient prefix their Radical Sign.

Examples.

1 ÷ 2	1	$\sqrt{ba}$	$\sqrt{bcaa + dcaa}$	$\sqrt{aaaa - bbbb}$
	2	$\sqrt{b}$	$\sqrt{ca}$	$\sqrt{aa - bb}$
	3	$\sqrt{a}$	$\sqrt{ba + da}$	$\sqrt{aa + bb}$

Case 2. If Surd Quantities, of the same kind, are joyned to Rational Quantities; then Divide the Rational, by the Rational if it can be, and to their Quotient, joyn the Quotient of the Surd, Divided by the Surd, with its first Radical Sign.

Examples.

1 ÷ 2	1	$3db\sqrt{bca}$	$15cda\sqrt{bcaa + dcaa}$	$75\sqrt{abd}$
	2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
	3	$d\sqrt{bc}$	$5cd\sqrt{ba + da}$	$15\sqrt{ab}$

Note.

Note. If any Square be Divided by its Root, the Quotient will be its Root.

Examples.

	1	a	$bb + 2bc + cc$	$aaaa - 2bbaa + bbbb$
	2	$\sqrt{a}$	$\sqrt{bb + 2bc + cc}$	$\sqrt{a^4 - 2bbaa + b^4}$
$1 \div 2$	3	$\sqrt{a}$	$\sqrt{bb + 2bc + cc}$	$\sqrt{a^4 - 2bbaa + b^4}$

SECT. 4. Involution of Surd Quantities.

Case 1. If the Surds are not joyn'd to Rational Quantities; they are Involved to the same height as their Index denotes, by only taking away their Radical Sign,

Examples.

	1	$\sqrt{a}$	$\sqrt{bca}$	$\sqrt{aa - bb}$	$\sqrt{5a - da}$
$1 \odot 2$	2	a	bca	$aa - bb$	$5a - da$

Case 2. When the Surds are joyned to Rational Quantities; Involve the Rational Quantities to the same height as the Index of the Surd denotes; then Multiply those Involved Quantities into the Surd Quantities, after their Radical Sign is taken away as before.

Examples.

	1	$b\sqrt{a}$	$5d\sqrt{ca}$	$3b\sqrt{aa - dd}$
$1 \odot 2$	2	bba	$25dda$	$9bbaa - 9bbdd$

	1	$a\sqrt[3]{bc}$	$3d\sqrt[3]{na + bb}$	$da\sqrt[3]{b}$
$1 \odot 3$	2	$aaabc$	$27dddaa + 27dddab$	$dddaaab$

The Reason of only taking away the Radical Sign, as in Case 1. is easily conceived, if you consider that any Root being Involved into it self, produces a Square, &c.

And from thence the Reason of those Operations performed by the Second Case, may be thus Stated.

Suppose $b\sqrt{a} = x$. Then $\sqrt{a} = \frac{x}{b}$ per Axiom 4. and both Sides of the Equation being equally Involved it will be $a = \frac{xx}{bb}$. Then Multiplying both Sides of the Equation into bb , it will become $bba = xx$ per Axiom 3. Which was to be proved.

Again

Again Let $\sqrt[5]{d} \sqrt{ca} = x$. Then $\sqrt{ca} = \frac{x}{\sqrt[5]{d}}$

And $ca = \frac{xx}{\sqrt[5]{d}}$ Consequently $\sqrt[5]{d} dca = xx$.

Also from hence it will be easie to deduce the Reason of *Multiplying Surd Quantities*, according to both the Cases. For

Suppose $\left\{ \begin{array}{l|l} 1 & \sqrt{b} = z \\ 2 & \sqrt{a} = x \end{array} \right\}$ Example 1. Case 1.
 $1 \odot 2$ 3 $b = zz$.
 $2 \odot 2$ 4 $a = xx$.
 3×4 5 $ba = zxxx$. per Axiom 3.
 $5 \sqrt{2}$ 6 $\sqrt{ba} = zx$. which was to be proved.

Let $\left\{ \begin{array}{l|l} 1 & d\sqrt{bc} = z \\ 2 & 3b\sqrt{a} = x \end{array} \right\}$ Example 1. Case 2.
 $1 \div d$ 4 $\sqrt{bc} = \frac{z}{d}$
 $2 \div 3b$ 5 $\sqrt{a} = \frac{x}{3b}$
 4×5 6 $\sqrt{bca} = \frac{zx}{3bd}$. from what is proved above.
 $6 \times 3bd$ 7 $3bd\sqrt{bca} = zx$, &c. for the rest.

Division being the converse to Multiplication, needs no other Proof.

CHAP. V.

Concerning the Nature of Equations, and how to prepare them for a Solution.

When any Problem or Question is proposed to be Analytically Resolved; it is very requisite that the true design or meaning thereof be fully and clearly comprehended (in all its parts) that so it may be truly abstracted from such Ambiguous words as Questions of this kind are often disguised with; otherwise it will be very difficult, if not impossible to state the Question aright in its Substituted Letters, and ever to bring it to an Equation, by such various Methods of ordering those Letters as the Nature of the Question may require.

Now

Now the knowledge of this difficult part of the work, is only to be obtained by Practice, and a careful minding the Solution of such Leading Questions as are in themselves very easie. And for that Reason I have inserted a collection of several Questions; wherein there is great variety.

Having got so clear an Understanding of the Question proposed, as to place down all the Quantities concerned in their due Order, viz. all the Substituted Letters, in such Order as the Nature of the Question requires; the next thing must be to consider whether it be Limited or not. That is, whether it admits of more Answers than one. And to discover that, observe the Two following Rules.

Rule 1. $\left\{ \begin{array}{l} \text{When the Number of the Quantities Sought, ex-} \\ \text{ceed the Number of the given Equations, the} \\ \text{Question is capable of innumerable Answers.} \end{array} \right.$

Example:

Suppose a Question were proposed thus; There are Three such Numbers, that if the First be Added to the Second, their Sum will be 22. And if the Second be Added to the Third, their Sum will be 46. What are those Numbers.

Let the Three Numbers be represented by Three Letters, thus, call the First a , the Second e , and the Third y .

Then $\left\{ \begin{array}{l} a + e = 22 \\ e + y = 46 \end{array} \right. \}$ according to the Question.

Here the Number of Quantities Sought are Three; viz. a , e , y , and the Number of the given Equations are but Two. Therefore, this Question is not Limited, but admits of various Answers; because for any one of those Three Letters, you may take any Number at pleasure that is Less than 22. Which with a little consideration will be very easie to conceive.

Rule 2. $\left\{ \begin{array}{l} \text{When the Number of the given Equations (not} \\ \text{depending upon one another) are just as many as} \\ \text{the Number of the Quantities Sought; Then is} \\ \text{the Question truly limited, viz. each Quantity} \\ \text{Sought hath but one single value.} \end{array} \right.$

As for instance, Let the foresaid Question be proposed thus. There are Three Numbers (a , e , and y , As before) if the First be Added to the Second, their Sum will be 22

if the *Second* be *Added* to the *Third*, their *Sum* will be 46; and if the *First* be *Added* to the *Third*, their *Sum* will be 36. What are the *Numbers*?

That is, $a + e = 22$. $e + y = 46$. and $a + y = 36$.

Now the *Question* is perfectly limited, each single *Quantity* having but one single *Value*, to wit $a = 6$. $e = 16$. and $y = 30$.

N. B. If the *Number* of the given *Equations*, exceeds the *Number* of the *Quantities* sought; they not only limit the *Question*, but oftentimes render it impossible, by being propos'd inconsistent one to another.

Having truly Stated the *Question* in its substituted *Letters*, and found it limited to one *Answer*, (or at least so bounded as to have a certain determinate *Number* of *Answers*) then let all those substituted *Letters* be so ordered or compared together, either by Adding, Subtracting, Multiplying or Dividing them, &c. according as the *Nature* of the *Question* requires, until all the unknown *Quantities* except One, are cast off or vanished; but therein great *Care* must be taken to keep them to an exact *Equality*; and when that unknown *Quantity*, or some *Power* of it (as *Square*, *Cube*, &c.) is found equal to those that are known; then the *Question* is said to be brought to an *Equation*, and consequently to a *Solution*, viz. fitted for an *Answer*.

But no particular *Rules* can be prescribed for the casting off, or getting away *Quantities* out of an *Equation*; that part of the *Art*, is only to be obtained by *Care* and *Practice*. And when that is done, it generally happens so, that the unknown *Quantity* which is retained in the *Equation*, is so mix'd and intangled with those that are known; that it often requires some *Trouble* and *Skill* to bring it (or its *Powers*, &c.) to one *Side* of the *Equation*, and those that are known to the other *Side*; (still keeping them to a just *Equality*) which the Ingenious *Van Schooten* in his *Principia Mathematicos Universalis* calls *Reduction* of *Equations*.

The *Business* of reducing *Equations* (as of most, if not all *Algebraick Operations*) is grounded and depends upon a right Application of the five *Axioms* proposed in Page 146. and therefore, if those *Axioms* be well understood, the *Reason* of such *Operations* must needs appear very plain, and the *Work* be easily perform'd; as in the following *Sections*.

Sect. 1. Of Reduction by Addition.

Reduction by Addition is grounded upon Axiom 1. and is only the Transposing (viz. the removing) of any Negative Quantity from either Side of an Equation to the other Side, with the Sign + before it; As in these

Examples.

Suppose	1	$a - b = d$	Again,	1	$aa - d = c - aa$
Then	2	$a = d + b$	Let	2	$aa = c - aa + d$
For	3	$b = b$	1 + d	3	$2aa = c + d$
1 + 3	4	$a = d + b$	2 + aa		

Let	1	$3a - 4 = 6 - a$	} Note, When any absolute Number is Register'd in the Margin; you must draw a Line over it, to distinguish it from the other Numbers. As 4 in the 2d Step of this Example.
1 + 4	2	$3a = 6 + 4 - a$	
2 + a	3	$4a = 6 + 4 = 10$	

Let	1	$aa - dc - b = dd - 2ba$
1 + b	2	$aa - dc = dd - 2ba + b$
2 + dc	3	$aa = dd - 2ba + b + dc$
3 + 2ba	4	$aa + 2ba = dd + b + dc$

Suppose	1	$2da - d = cc - 3baa - aaa$
1 + aaa	2	$aaa + 2da - d = cc - 3baa$
2 + 3baa	3	$aaa + 3baa + 2da - d = cc$
4 + d	4	$aaaa + 3baa + 2da = cc + d$ &c.

Sect. 2. Of Reduction by Subtraction.

Reduction by Subtraction is grounded upon Axiom 2. and is perform'd by Transposing (or removing) any Affirmative Quantity from either side of the Equation, to the other side, with the Sign - before it. As in these

Examples.

Suppose	1	$a + b = d$	Let	1	$3a + 4 = 6 + a$
And	2	$b = b$	1 - a	2	$2a + 4 = 6$
1 - 2	3	$a = d - b$	2 - 4	3	$2a = 6 - 4 = 2.$

Suppose	1	$aa + dc + b = dd + 2ba$
1 - 2ba	2	$aa - 2ba + dc + b = dd$
2 - dc	3	$aa - 2ba + b = dd - dc$
3 - b	4	$aa - 2ba = dd - dc - b$

Let

Let	1	$aaa + d = cc + 3baa + 2da$
1 — 3baa	2	$aaa - 3baa + d = cc + 2da$
2 — 2da	3	$aaa - 3baa - 2da + d = cc$
3 — d	4	$aaa - 3baa - 2da = cc - d$

Sect. 3. Of Reduction by Multiplication.

Fractional Quantities in any Equation, are brought into whole Quantities; by Multiplying every Term in the Equation with the Denominators of the Fractions. per Axiom. 3. As in these

Examples.

Suppose 1 $\left| \frac{a}{5} = 6 \right.$

Then 2 $\left| a = 6 \times 5 = 30. \text{ For } \frac{a}{5} \times 5 = \frac{5a}{5} = a \right.$

Let	1	$3a = \frac{dc}{2b}$	Suppose	1	$a = \frac{dd}{a-b}$
1 $\times 2b$	2	$6ba = dc$	1 $\times a - b$	2	$aa - ba = dd$

Suppose	1	$\frac{aa}{b} + c + f = \frac{dx}{a}$
1 $\times b$	2	$aa + bc + bf = \frac{dxb}{a}$
2 $\times a$	3	$aaa + bca + bfa = dxb$

Suppose	1	$\frac{aaa}{aa-bb} = \frac{ba-bb}{a+b}$
1 $\times aa - bb$	2	$aaa = \frac{baaa - bbba - bbba - bbbb}{a+b}$
1 $\times a + b$	3	$aaaa + baaa = baaa - bbba - bbba + bbbb$

Sect. 4. Of Reduction by Division.

When any Quantity (either known, or unknown) is in every Term of an Equation; if the whole Equation, be Divided by that Quantity, it will be Reduc'd into Lower Terms. per Axiom 4. As in these

Examples.

$$\begin{array}{l|l|l} \text{Suppose} & 1 & baa + bca = bcd \\ 1 \div b & 2 & aa + ca = cd \end{array} \quad \begin{array}{l|l|l} \text{Let} & 1 & aa = 7a \\ 1 - a & 2 & a = 7 \end{array}$$

$$\begin{array}{l|l|l} \text{Let} & 1 & ffaa + ffca - ffa = ffa + ffa \\ 1 \div ff & 2 & aa + ca - a = da + dda \\ 2 \div a & 3 & a + ca - 1 = d + dd \end{array}$$

Or when the *unknown Quantity*, is *Multiplied* (*viz. joyn'd*) with any that is *known*; let the whole *Æquation* be *Divided* by the *known Quantity*, that so the *unknown* may be *cleared*. As in these

Examples.

$$\begin{array}{l|l|l} \text{Suppose} & 1 & ba - ca = d \\ 1 \div b - c & 2 & a = \frac{d}{b - c} \end{array} \quad \begin{array}{l|l|l} \text{Let} & 1 & caa - daa = cd - dd \\ 1 \div c - d & 2 & aa = \frac{cd - dd}{c - d} = d \end{array}$$

$$\begin{array}{l|l|l} \text{Suppose} & 1 & bbaa - 2bba = bda + cba \\ 1 \div ba & 2 & baa - 2ba = d + c \\ 2 \div b & 3 & aa - 2a = \frac{d + c}{b} \end{array}$$

$$\begin{array}{l|l|l} \text{Let} & 1 & 49daa + 42aa = 7bca + 21ca \\ 1 \div 7 & 2 & 7daa + 6aa = bca + 3ca \\ 2 \div a & 3 & 7da + 6a = bc + 3c \\ 3 \div & 4 & a = \frac{bc + 3c}{7d + 6} \end{array}$$

Sect. 5. Of Reduction by Involution.

When there happens to be an *Æquation*, between any *Homogenous* or like *Surds*; Take away the *Radical Signs* from the *Quantities*, and they will become *Rational*. As in these

Examples.

$$\begin{array}{l|l|l} \text{Suppose} & 1 & \sqrt{a} = \sqrt{d + c} \\ 1 \odot 2 & 2 & a = d + c \end{array} \quad \begin{array}{l|l|l} \text{Let} & 1 & \sqrt[3]{aa} = \sqrt[3]{db + bc} \\ 1 \odot 3 & 2 & aa = db + bc. \end{array} \left. \begin{array}{l} \text{per Sect.} \\ \text{4. Ch. 3.} \end{array} \right\}$$

Or if one *Side* of the *Æquation* consists of *Surd Quantities*, and the other *Side* be *Rational*; Then *Involve* the *Rational Quantity*.

Quantities to the same Power (or height) with the Index of the Surd, and take away the Radical Sign. As in these

Examples.

Let	1	$\sqrt{a} = 6$	Suppose	1	$\sqrt{a} = b + c$
1 ⊕ 2	2	$a = 36$	1 ⊕ 2	2	$a = bb + 2bc + cc.$

Suppose	1	$\sqrt[3]{aa - ba} = d$	Let	1	$\sqrt[3]{aa} = 7$
1 ⊕ 3	2	$ca - ba = ddd$	1 ⊕ 5	2	$aa = 16807.$

SECT. 6. Of Reduction by Evolution.

When any Single Power of the unknown Quantity is at one Side of an Equation; Evolve both Sides of the Equation, according as the Index of that Power denotes, and their Roots will be equal. As in these

Examples.

Suppose	1	$aa = 36$	Let	1	$aaa = 27$
1 ∪ 2	2	$a = \sqrt{36} = 6$	1 ∪ 2	2	$a = \sqrt[3]{27} = 3, \&c.$

Suppose	1	$aa = bb - dd$	Let	1	$aaa = b^3 + 3bbc + 3bcc + c^3$
1 ∪ 2	2	$a = \sqrt{bb - dd}$	1 ∪ 3	2	$a = b + c$

Or if any Compound Power of the unknown Quantity, be at one Side of the Equation (that hath a true Root of its kind) Evolve both Sides of the Equation, and it will be deprefs'd into Lower Terms. As in these

Examples.

Suppose	1	$aa + 2ba + bb = dd$	$aa - 2ba + bb = ddcc$
1 ∪ 2	2	$a + b = d$	$a - b = dc.$

Here follows a few Examples of Clearing Equations, wherein all the foregoing Reductions are promiscuously used, as occasion requires.

Example 1.

Suppose	1	$\frac{aa + c - d}{4} = \frac{g - aa}{b}.$	What is $a =$ To
1 × 4	2	$aa + c - d = \frac{4g - 4aa}{b}$	

2 × b

$a \times b$	3	$baa + bc - bd = 4g - 4aa$
$3 + 4aa$	4	$baa + 4aa + bc - bd = 4g$
$4 + bd$	5	$baa + 4aa + bc = 4g + bd$
$5 - bc$	6	$baa + 4aa = 4g + bd - bc$
$6 \div b + 4$	7	$aa = \frac{4g + bd - bc}{b + 4}$
$7 \text{ uw } 2$	8	$a = \sqrt{\frac{4g + bd - bc}{b + 4}}.$ As was requir'd.

Example 2.

Suppose	1	$\frac{a + 354}{a} = \frac{3a}{354 - a}$	What is the Value of a ,
$1 \times a$	2	$a + 354 = \frac{3aa}{354 - a}$	
$2 \times 354 - a$	3	$125316 - aa = 3aa$	
$3 + aa$	4	$4aa = 125316$	
$4 \div 4$	5	$aa = 31329$	
$5 \text{ uw } 2$	6	$a = \sqrt{31329} = 177$	the Value of a requir'd.

Example 3.

Suppose	1	$\sqrt{\frac{aa + 3bb}{4}} - \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}} \quad a = ?$
$1 \odot 2$	2	$\left\{ \begin{array}{l} \frac{aa + 3bb}{4} : - 2 \sqrt{\frac{aa + 3bb}{4}} \times \sqrt{\frac{aa - 3bb}{4}} \\ : + \frac{aa - 3bb}{4} = \frac{baa}{c} \end{array} \right.$
That is	3	$\frac{aa}{2} - \sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c}$
For		$\frac{aa + 3bb}{4} + \frac{aa - 3bb}{4} = \frac{2aa}{4} = \frac{aa}{2}$
And		$2 \sqrt{\frac{aa + 3bb}{4}} = \sqrt{\frac{4aa + 12bb}{4}} = \sqrt{aa + 3bb}$
Then		$\sqrt{aa + 3bb} \times \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{a^4 - 9b^4}{4}}$
$3 + \sqrt{\&c.}$	4	$\frac{aa}{2} = \frac{baa}{c} + \sqrt{\frac{a^4 - 9b^4}{4}}$

4 - $\frac{baa}{c}$	5 $\frac{aa}{2} - \frac{baa}{c} = \sqrt{\frac{a^4 - 9b^4}{4}}$
5 $\odot$ 2	6 $\frac{a^4}{4} - \frac{ba^4}{c} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4}$
6 + $\frac{ba^4}{c}$	7 $\frac{a^4}{4} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4} + \frac{ba^4}{c}$
7 -	8 $\frac{bba^4}{cc} + \frac{9b^4}{4} = \frac{ba^4}{c}$
8 $\div b$	9 $\frac{ba^4}{cc} + \frac{9b^3}{4} = \frac{a^4}{c}$
9 $\times cc$	10 $ba^4 + \frac{9ccb^3}{4} = ca^4$
10 $\times \frac{4}{4}$	11 $4ba^4 + 9ccb^3 = 4ca^4$
11 - $4ba^4$	12 $9ccb^3 = 4ca^4 - 4ba^4$
12 $\div$	13 $aaaa = \frac{9ccb^3}{4c - 4b}$
For	$4c - 4b \times a^4 = 4ca^4 - 4ba^4$
13 uw 2	14 $aa = \sqrt{\frac{9ccb^3}{4c - 4b}}$
14 uw 2	15 $a = \sqrt{\frac{9ccb^3}{4c - 4b}}$ As was requir'd.

By help of these *Reductions* (properly applied) the *unknown Quantity* (a) or its *Powers*, are cleared and brought to one *Side* of an *Equation*; And if the *unknown Quantity* (a) chance to be equal to those that are *known*, the *Question* is *Answered*; as in the first *Examples* of *Seet. 1. and 2.*

Or if any *Single Power* of the *unknown Quantity* (a) is found *Equal* to those that are *known*, then the *respective Root* of the *known Quantities*, is the *Answer*; as in the first 4 *Examples* of *Seet. 6. &c.*

But when the *Powers* of the *unknown Quantity*, are either mixed with their *Root*; As $aa + ba = dd$, &c. Or do consist of different *Powers*; As $aaa + baa = dd$, &c. Then they are called *Affected*, or *Adaffected Equations*, which require other *Methods* to *Resolve* them, *viz.* to find out the *Value* of (a) as shall be shewed further on.

C H A P. VI.

Of Proportional Quantities; both Arithmetical, Geometrical and Musical.

What hath been said of Numbers in *Arithmetical Progression*, Chap. 6. Part I. may be easily Applied to any Series of Homogenous or Like Quantities.

Sect. I. Of Quantities in Arithmetical Progression.

Those Quantities are said to be in the most Simple or Natural Progression, that begin their Series of Increase, or Decrease with a Cypher.

Thus $\begin{cases} 0 : a : 2a : 3a : 4a : 5a : 6a : & \&c. \text{ Increasing} \\ 0 : -a : -2a : -3a : -4a : -5a : -6a : & \&c. \text{ Decreasing} \end{cases}$

Or Universally, putting a the first Term in the Progression, and e , the common Excess or Difference.

Then $\begin{cases} a : a + e : a + 2e : a + 3e : a + 4e : a + 5e : a + 6e : & \&c. \\ a : a - e : a - 2e : a - 3e : a - 4e : a - 5e : a - 6e : & \&c. \end{cases}$

In the first of these Series it's evident, that if there be but Three Terms; the Sum of the Extremes will be Double to the Mean.

As in these, $0 : a : 2a$: Or, $a : 2a : 3a$: Or, $2a : 3a : 4a$, &c. viz. $2a + 0 = a + a$: Or, $a + 3a = 2a + 2a$, &c.

Also in the Second Series, either Increasing or Decreasing, it's evident, that if the Terms be $a : a + e : a + 2e$, &c. Increasing; Then $a + a + 2e$, viz. $2a + 2e$ the Sum of the Extremes, is double to $a + e$ the Mean; Or if they be $a : a - e : a - 2e$, &c. Decreasing, then $a + a - 2e$, viz. $2a - 2e$ the Sum of the extremes, is Double to $a - e$, the Mean. And so it will be in any other Three of the Terms. Secondly, If there are Four Terms; then the Sum of the Two Extremes, will be equal to the Sum of the Two Means. As in these, $a : a + e : a + 2e : a + 3e$, in the Series Increasing; Here $a + a + 3e = a + e + a + 2e$.

Also in these, $a : a - e : a - 2e : a - 3e$, in the Series Decreasing; here $a + a - 3e = a - e + a - 2e$, &c. in any other four Terms.

Consequently, If there are never so many Terms in the Series, the Sum of the Two extremes, will always be equal to the Sum

of any *Two Means*, that are *equally distant* from those *Extreames*.
 As in these, $a : a + e : a + 2e : a + 3e : a + 4e : a + 5e$, &c.
 Here $a + a + 5e = a + e + a + 4e = a + 2e + a + 3e$, &c.
 And if the *Number of Terms* be *Odd*; the *Sum* of the *Two Extreames* will be *Double* to the *Middle Term*, &c. As in Coroll.
 1. Chap. 6. beforemention'd.

Confectary 1.

Whence it follows (and is very easie to conceive) that if the Sum of the two Extreames be multiplied into the Number of all the Terms in the Series, that Product will be Double the Sum of all the Series.

Now for the easier Resolving such Questions as depend upon these Progressional Quantities,

Let $\begin{cases} a = \text{the first Term, as before.} \\ y = \text{the last Term.} \\ e = \text{the Common Excess, \&c. as before.} \\ N = \text{the Number of all the Terms.} \\ S = \text{the Sum of all the Series, viz. of all the Terms.} \end{cases}$

Then will $a + y \times N = 2S$, by the precedent Confectary :

That is $Na + Ny = 2S$. Consequently $\frac{Na + Ny}{2} = S$, the

Sum of all the Series, be the Terms never so many. Thirdly, In these Series, it is easie to perceive, that the common Difference (e) is so often Added to the Last Term of the Series ; As are the Number of Terms, except the First ; That is, the first Term (a) hath no Difference Added to it, but the Last Term hath so many times (e) Added to it, as it is distant from the First.

Consequently, the Difference betwixt the Two Extreames, is only the common Difference (e) Multiplied into the Number of all the Terms Less Unity or 1.

That is, $N - 1 \times e = y - a$ the Difference betwixt the two Extreames, viz: $Ne - e = y - a$.

Confectary 2.

Whence it follows, that if the Difference betwixt the Two Extreames be divided by the Number of Terms Less 1. the Quotient will be the common Difference of the Series.

To Wit. $\frac{y - a}{N - 1} = e$

Now by help of these *Two Confectaries*, if any *Three* of the aforeſaid *Five Parts* (*viz.* *a. y. e. N. S.*) be given; the other *Two* may be *eafily* found.

Thus,	1	$\frac{Na + Ny}{2} = S$	} <i>As before.</i>
And	2	$\frac{y - a}{N - 1} = e$	
$2 \times N - 1$	3	$y - a = Ne - e$	
$3 + e$	4	$y - a + e = Ne$	
$4 \div e$	5	$\frac{y - a + e}{e} = N$	<i>The Number of Terms.</i>
1×2	6	$Na + Ny = 2S$	
$6 - Na$	7	$Ny = 2S - Na$	
$7 \div N$	8	$\frac{2S - Na}{N} = y$	<i>The laſt Term.</i>
$6 - yN$	9	$Na = 2S - Ny$	
$9 \div N$	10	$\frac{2S - Ny}{N} = a$	<i>The firſt Term.</i>
$6 \div a + y$	11	$\frac{2S}{a + y} = N$	<i>The Number of Terms.</i>
5, and 11	12	$\frac{y - a + e}{e} = \frac{2S}{a + y}$	<i>Per Axiom 5.</i>
$12 \times a + y$	13	$\frac{yy - aa}{e} + a + y = 2S$	
$13 \div 2$	14	$\frac{yy - aa}{2e} + \frac{a + y}{2} = S$	<i>The Sum of all the Series.</i>
$14 \times 2e$	15	$yy - aa + ae + ye = 2Se$	
$15 - ae$	16	$yy - aa + ye = 2Se - ae$	
$16 - ye$	17	$yy - aa = 2Se - ae - ye$	
$17 \div$	18	$\frac{yy - aa}{2S - a - y} = e$	<i>The common Difference.</i>
$3 + a$	19	$Ne - e + a = y$	<i>The laſt Term.</i>
$19 + e$	20	$Ne + a = y + e$	
$20 - Ne$	21	$y + e - Ne = a$	<i>The Firſt Term.</i>
			&c.

In like manner you may *proceed* to find out any of the five *Quantities* (*a. e. y. N. S.*) otherways, *viz.* by *Varying* and *Comparing* of theſe *Equations* one with another, you may *produce* new *Equations* with other *Data* in them; the which I ſhall

I shall here omit pursuing, and leave them for the Learner's Practice.

SECT. 2. Of Quantities in Geometrical Proportion.

Geometrical Proportion continued has been already Defined in SECT. 2. Chap. 6. Part. 1. And what is there said concerning Numbers in $\therefore$ may easily be applied to any sort of Homogeneous Quantities that are in $\therefore$.

The most Natural and Simple Series of Geometrical Proportionals, is, when it begins with Unity or 1.

As 1 . a . aa . aaa . aaaa . a⁵ . a⁶ , &c. in $\therefore$

For 1 : a :: a : aa :: aa : aaa :: aaa : aaaa , &c

Or a . b . $\frac{bb}{a}$. $\frac{bbb}{aa}$. $\frac{bbbb}{aaa}$. $\frac{b^5}{a^4}$; &c. are Terms in $\therefore$.

For a ; b :: b : $\frac{bb}{a}$:: $\frac{bb}{a}$: $\frac{bbb}{aa}$:: $\frac{bbb}{aa}$: $\frac{b^4}{a^3}$:: $\frac{b^4}{a^3}$: $\frac{b^5}{a^4}$. &c

That is, when all the middle Terms betwixt the two Extremes are both Consequents and Antecedents, that Series is in Geometrical Proportion continued.

Therefore in every Series of Quantities in $\therefore$ all the Terms except the Last are Antecedents; and all the Terms except the First are Consequents.

But Universally, putting a the first Term in the Series, and e the Ratio, viz. the common Multiplier or Divisor, then the will be

a . ae . aee . aeee . aeeee . ae⁵ . ae⁶ . &c. in $\therefore$

Or a . $\frac{a}{e}$. $\frac{a}{ee}$. $\frac{a}{eee}$. $\frac{a}{e^4}$. &c. are in $\therefore$ Decr.

For a : ae :: ae : $\frac{aee}{a}$ = aee &c.

And a : $\frac{a}{e}$:: $\frac{a}{e}$: $\frac{aa}{aee}$ = $\frac{a}{ee}$ a : $\frac{a}{e}$:: $\frac{a}{ee}$: $\frac{a}{eee}$ &c.

I. In any of these Series it is Evident, that if three Quantities are in $\therefore$ the Rectangle of the two Extremes will be equal to the Square of the Mean. As in these.

a : ae . aee . here a x aee = ae x ae . = aeee . &c.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee}$ here also, $a \times \frac{a}{ee} = \frac{a}{e} \times \frac{a}{e} = \frac{aa}{ee}$ &c.

II. If Four Quantities are in $\therefore$ the Rectangle of the Extreames will be equal to the Rectangle of the Means.

As in these, $a \cdot ae \cdot aee \cdot aeee$. here $a \times ae^3 = ae \times aee$.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee}$ here also $a \times \frac{a}{eee} = \frac{a}{e} \times \frac{a}{ee} = \frac{aa}{eee}$ &c.

Consequently, If there are never so many Terms in the Series of $\therefore$ the Rectangle of the Extreames will be equal to the Rectangle of any Two Means that are equally distant from those Extreames.

As in these, $a \cdot ae \cdot aee \cdot aeee \cdot aeeee \cdot ae^5$ &c.

viz. $ae^5 \times a = ae^4 \times ae$ Or $ae^5 \times a = aeee \times aee = ae^5$.

III. If never so many Quantities are in $\therefore$ it will be, As any one of the Antecedents, Is to its Consequent; So is the Sum of all the Antecedents, To the Sum of all the Consequents.

As in these, $\left\{ \begin{array}{l} a \cdot ae \cdot aee \cdot aeee \cdot aeeee \cdot ae^5, \text{ \&c. Increasing} \\ a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee} \cdot \frac{a}{eeee} \cdot \frac{a}{e^5} \text{ \&c. Decreasing.} \end{array} \right.$

$a : ae :: a + ae + aee + ae^3 + ae^4 : ae + aee + ae^3 + ae^4 + ae^5$

Or $a : \frac{a}{e} :: a + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} : \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} + \frac{a}{e^5}$

viz. $a \times ae + aee + ae^3 + ae^4 + ae^5 = ae \times a + ae + aee + ae^3 + ae^4$.

That is, the Rectangle of the Extreames, is equal to the Rectangle of the Means; per Second of this Sect.

Note, the Ratio of any Series in $\therefore$ increasing, is found by Dividing any of the Consequents by its Antecedent.

Thus, $a) ae (e$ Or $ae) aee (e$ &c.

But if the Series be decreasing, then the Ratio is found by Dividing any of the Antecedents by its Consequent.

Thus, $\frac{a}{e}) a (e$ Or $\frac{a}{ee}) \frac{a}{e} (e$ &c.

Confectary.

These things being premised, such *Æquations* may be deduced from them, as will solve all such Questions as are usually proposed about Quantities in Geometrical Proportion $\therefore$ In order to that

Let $\left\{ \begin{array}{l} a = \text{The first Term.} \\ e = \text{The common Ratio.} \\ y = \text{The Last Term.} \\ S = \text{The Sum of all the Terms.} \end{array} \right\}$ as before.

Then $S - y =$ the Sum of all the Antecedents.
And $S - a =$ the Sum of all the Consequents.

Analogy.	1	$a : ae :: S - y : S - a.$	Per the III. of this S. &.
1 $\therefore$	2	$Sa - aa = aeS - aey$	
2 $\div a$	3	$S - a = eS - ey$	
3 $+ ey$	4	$S + ey - a = eS$	
4 $- S$	5	$ey - a = eS - S$	
5 $\div e - 1$	6	$\frac{ey - a}{e - 1} = S$	The Sum of all the Series
3 $\div S - y$	7	$\frac{S - a}{S - y} = e$	The common Ratio.
5 $+ a$	8	$ey = eS + a - S$	
8 $\div e$	9	$\frac{eS + a - S}{e} = y$	The last Term.
4 $+ a$	10	$S + ey = eS + a$	
10 $- eS$	11	$S + ey - eS = a$	The First Term.

Note, The $\therefore$ set in the Margin at the second Step, is instead of Ergo; and signifies that the Rectangle of the two Extreames in the first Step, is equal to the Rectangle of the Means. And so for any other Proportion.

Señ. 3. Of Harmonical Proportion.

Harmonical or Musical Proportion is, when of three Quantities (or rather Numbers) the First hath the same Ratio to the Third, As the Difference between the First and Second, hath to the Difference between the Second and Third. As in these following.

Suppose a, b, c in Musical Proportion.

Then $\left\{ \begin{array}{l} 1 \quad a : c :: b - a : c - b \\ 2 \quad cb - ca = ac - ba \end{array} \right.$

2 $+ ca$

$2 + ca$	3	$cb = 2ac - ba$
$3 \div 2c - b$	4	$\frac{cb}{2c - b} = a$ The first Term.
$3 + ba$	5	$2ac = cb + ba$
$5 \div c + a$	6	$\frac{2ac}{c + a} = b$ The second Term.
$5 - cb$	7	$2ac - cb = ba$
$7 \div 2a - c$	8	$\frac{ba}{2a - c} = c$ The third Term.

If there are *Four Terms* in *Musical Proportion*, the *First* hath the same *Ratio* to the *Fourth*, as the *Difference* between the *First* and *Second*, hath to the *Difference* between the *Third* and *Fourth*. That is, Let *a. b. c. d* be the *four Terms*, &c.

Then	1	$a : d :: b - a : d - c$
1 $\therefore$	2	$db - da = da - ca$
$2 + da$	3	$db = 2da - ca$
$3 \div 2d - c$	4	$\frac{db}{2d - c} = a$
$3 \div d$	5	$b = \frac{2da - ca}{d}$
$3 + ca$	6	$db + ca = 2da$
$6 - db$	7	$ca = 2da - db$
$7 \div a$	8	$c = \frac{2da - db}{a}$
$7 \div 2a - b$	9	$\frac{ca}{2a - b} = d$

CHAP. VII.

Of Proportion Disjunct, and how to turn *Equations* into *Analogies*, &c.

Proportion Disjunct, or the *Rule of Three* in *Numbers*, is already explain'd in *Chap. 7. Part. 1.* And what hath been there said, is applicable to all *Homogeneous Quantities*, viz. of *Lines* to *Lines*, &c.

Sect. I.

If *Four Quantities* (viz. either *Lines*, *Superficies*, or *Solids*) be *Proportional*; the *Rectangle* comprehended under the *Extremes*,

is equal to the Rectangle comprehended under the Two Means, 16. Euclid 6.

For instance, Suppose, $a . b . c . d$. to represent the Four Homogeneous Quantities in Proportion,

viz. $a : b :: c : d$. Then will $ad = bc$

For suppose $b = 2a$ then will $d = 2c$

And it will be $a : 2a :: c : 2c$. Here the Ratio is 2.

but $a \times 2c = 2a \times c$. viz. $2ca = 2ac$

Or Suppose $b = 3a$ then will $d = 3c$

And it will be $a : 3a :: c : 3c$. Here the Ratio is 3.

but $a \times 3c = 3a \times c$ viz. $3ac = 3ac$

Or Universally putting e for the Ratio of the Proportion, viz. making $b = ae$, then will $d = ce$

And it will be $a : ae :: c : ce$

but $a \times ce = ae \times c$ viz. $ace = aec$

Consequently, $ad = bc$ which was to be proved.

Whence it follows, that if any Three of the four Proportional Quantities be given, the fourth may be easily found. Thus,

Let	1	$a : b :: c : d$.	Then
1 $\therefore$	2	$ad = bc$	as before
		$\frac{bc}{d}$	
$2 \div d$	3	$a = \frac{bc}{d}$	
		$\frac{ad}{c}$	
$2 \div c$	4	$b = \frac{ad}{c}$	
		$\frac{ad}{b}$	
$2 \div b$	5	$c = \frac{ad}{b}$	
		$\frac{bc}{a}$	
$2 \div a$	6	$d = \frac{bc}{a}$	
		$\frac{a}{b} = \frac{c}{d}$	Note, In this manner Euclid, in his 5th Book, expresses the Ratio of Proportionals. viz. the Ratio of a To b Is $\frac{a}{b}$ &c.
$2 \div bd$	7	$\frac{b}{a} = \frac{d}{c}$	
Or $2 \div ac$	8	$\frac{b}{a} = \frac{d}{c}$	

If four Quantities are Proportional, they will also be Proportional in Alternation, Inversion, Composition, Division, Conversion, and Mixtly. Euclid. 5. Def. 12, 13, 14, 15, 16.

That

That is, if	1	$a : b :: c : d$ be in direct Proportion, as before.
Then	2	$a : c :: b : d$ Alternate. For $ad = bc$
And	3	$b : a :: d : c$ Inverted. For $ad = bc$
Also	4	$a + b : b :: c + d : d$ Compounded.
4 $::$	5	$da + bd = bc + bd$ That is, $ad = bc$, as before.
Or	6	$a + c : c :: b + d : d$ Alternately Compounded.
6 $::$	7	$ad + cd = bc + cd$ That is, $ad = bc$.
Again,	8	$a - b : b :: c - d : d$ Divided.
8 $::$	9	$ad - bd = bc - bd$ That is, $ad = bc$.
Or	10	$a - c : c :: b - d : d$ Alternately Divided.
10 $::$	11	$ad - cd = bc - cd$ That is, $ad = bc$.
And	12	$a : b \pm a :: c : d \pm c$ Converted.
12 $::$	13	$ad \pm ac = bc \pm ac$ That is, $ad = bc$.
Lastly	14	$a + b : a - b :: c + d : c - d$ Mixtly.
14 $::$	15	$ac - ad + bc - bd = ac - ad - bc - bd$.
15 $\pm$	16	$2bc = 2ad$; That is, $ad = bc$ As at first.

Note, What has been here done about whole Quantities in Simple Proportion, may be easily perform'd in Fractional Quantities; And Surds, &c.

For Instance, If $\frac{ab}{c} : \frac{d-c}{f} :: \frac{d+c}{o}$ and it be required to find the fourth Term,

it will be $\frac{dd - cc}{fc}$ the Rectangle of the Means, which being

Divided by the first Extreame $\frac{ab}{c}$ it will become

$$\frac{ab}{c}) \frac{dd - cc}{fc} (\frac{ddc - ccc}{abfc} = \frac{dd - cc}{abf} \text{ the fourth Term}$$

Or if $b : \sqrt{bd + bc} :: \sqrt{bd + bc} : \text{to a fourth Term,}$

Then is $\sqrt{bd + bc} \times \sqrt{bd + bc} = bd + bc$ the Rectangle of The Means, And $b) bd + bc (d + c$ the fourth Term.

That is, $b : \sqrt{bd + bc} :: \sqrt{bd + bc} : d + c$ &c.

SECT. 2. Of DupliCAT and TriPLiCAT Proportion.

The Proportions treated of in the last Section, are to be understood only when Lines are compared to Lines, and Superficies to Superficies.

Superficies; or *Solids* to *Solids*; viz. when each is compared to that of its like kind, which is only called *Simple Proportion*.

But when *Lines* are compared to *Superficies*, or *Lines* are compared to *Solids*, such Comparisons are distinguished from the former, by the Names of *Duplicate* and *Triplicate* (&c.) *Proportions*; so that *Simple*, *Duplicate* and *Triplicate* (&c.) *Proportions*, are to be understood in a different Sence from *Single*, *Double*, *Treble*, &c. *Proportions*, which are only as 1, 2, 3, &c. to 1; but those of *Simple*, *Duplicate*, *Triplicate*, &c. *Proportions*, is that of $a : aa : aaa$, &c. to 1. Or if the *Simple Proportions* be that of a to b , whose Ratio or Exponent is $\frac{a}{b}$ or $\frac{b}{a}$ according to *Euclid's* way.

Then $\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb}$ is the Exponent of the Duplicate.

And $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$ is the Exponent of the Tripl. Propor.

&c.

And if there are three, four, or more Quantities in $a : a^2 : a^3 : a^4 : a^5$, &c. (As in the first Series Sect. 2. of the last Chapter.) Then, that of the First to the Third, Fourth and Fifth, &c. (viz. 1 To aa , aaa , $aaaa$, a^5) is *Duplicate*, *Triplicate*, *Quadruplicate*, &c. of the First to the Second (viz of 1 To a ;) And by *Inversion*, that of the Third, Fourth, Fifth, &c. is *Duplicate*, *Triplicate*, &c. of that of the Second to the First, (a To 1) per Diff. 10. *Eucl.* 5. But the Nature of these *Proportions* will appear more *Evident*, and be easier understood when they are applied to Practice, and Illustrated by *Geometrical Figures* further on.

Sect. 3. How to turn Equations into Analogies.

From the first Section of this Chap. it will be easie to conceive, how to Turn or dissolve *Equations* into *Analogies* or *Proportions*.

For if the *Rectangle* of two (or more) Quantities, be equal to the *Rectangle* of two (or more) Quantities; then are those four (or more) Quantities *Proportional*. By the 16 *Euclid.* 6.

That is if $ab = dc$, Then is $a : c :: d : b$.

Or $ac : ad :: ab : dc$ &c.

From whence there arises this general Rule for Turning *Equations* into *Analogies*.

C c

Rule

Rule { Divide either Side of the given Equation (if it can be done) into two such Parts, or Factors, as being multiplied together, will produce that Side again; And make those two Parts the two Extreames. Then Divide the other Side of the Equation (if it can be done) in the same Manner as the First was, and let those two Parts or Factors, be the two Means.

For Instance, Suppose $ab + ad = bd$.

Then $a : b :: d : b + d$. Or $b : a :: b + d : d$ &c.

Or taking ad from both Sides of the Equation, and

It will be $ab = bd - ad$. Then $a : d :: b - a : b$.

Or, $b : d :: b - a : a$ &c.

Again, suppose $aa + 2ae = 2by + yy$.

Here a and $a + 2e$ are the two Factors of the first Side in this Equation; for $a + 2e \times a = aa + 2ae$.

Again, y and $2b + y$ are the two Factors of the other Side

Therefore, $a : y :: 2b + y : a + 2e$

Or $2b + y : a + 2e :: a : y$ &c.

When one Side of any Equation can be divided into two Factors, as before; and the other Side can not be so Divided, then make the Square Root of that Side either the two Extreames or the two Means.

For Instance, Suppose $bc + bd = da + g$.

Then $b : \sqrt{da + g} :: \sqrt{da + g} : c + d$.

Or $\sqrt{da + g} : b :: c + d : \sqrt{da + g}$ &c.

C H A P. VIII.

Of Substitution, and the Solution of Quadratic Equations.

Sect. I. Of Substitution.

When new Quantities, not concerned in the First stating any Question, are put instead of some that are engag'd in that is called Substitution.

For Instance, If instead of $\sqrt{bc - dc}$ you put z , or any other Letter.

That is, make $z = \sqrt{bc - dc}$.

Or Suppose $aa - ba - ca + da = dc$, instead of $b - c - d$ you put s , or any other Letter not engaged, viz. $s = b - c - d$. Then $aa - sa = dc$.

That is, if c be greater than $b + d$, it's $aa - sa = dc$.

But if $b + d$ be greater than c , Then it's $aa + sa = dc$.

And this way of Substituting or putting of new Quantities instead of others, may be found very Useful upon several Occasions; viz. in order to render some following Operations in the Question more easie, and perhaps much shorter than they would be without it, as you may observe in some Questions hereafter proposed in this Treatise.

And when those Operations, in which the Substituted Quantities were assisting or useful, are performed according as the Nature of the Question required, you may then (if there be Occasion) bring the Original or first Quantities into the Equation, in the Place (or Places) of those Substituted Quantities: Which is called Restitution, as you will see further on.

Sect. 2. The Solution of Quadratick Equations.

When the Quantity sought is brought to an Equality with those that are known, and is at one Side of the Equation, in no more than two different Powers whose Indices are double one to another, those Equations are called Quadratick Equations Affected; and do fall under the Consideration of three Forms or Cases.

$$\begin{array}{l} \text{Case 1. } aa + 2ba = dc. \\ \text{Case 2. } aa - 2ba = dc. \\ \text{Case 3. } 2ba - aa = dc. \end{array} \left. \vphantom{\begin{array}{l} \text{Case 1.} \\ \text{Case 2.} \\ \text{Case 3.} \end{array}} \right\} \text{ And } \left\{ \begin{array}{l} a^4 + 2ba^2 = dc, \\ a^4 - 2ba^2 = dc, \\ 2ba^2 - a^4 = dc, \end{array} \right.$$

$$\text{Also } \left\{ \begin{array}{l} a^6 + 2ba^3 = dc. \\ a^6 - 2ba^3 = dc. \\ 2ba^3 - a^6 = dc. \end{array} \right\} \text{ And } \left\{ \begin{array}{l} a^8 + 2ba^4 = dc. \\ a^8 - 2ba^4 = dc. \\ 2ba^4 - a^8 = dc. \end{array} \right\} \text{ \&c.}$$

When there happens to be more Terms in one of these kind of Equations than Two, and the highest Power of the unknown Quantity is Multiplied into some known Co-efficients; you must Reduce them by Division; as in Sect. 4. of Chap. 5. and for the Fractional Quantities that may arise by those Divisions, Substitute another Quantity doubled.

For Instance, Let $baa + caa - ca - da = dc + cb$.

$$\text{Then } aa - \frac{ca - da}{b + c} = \frac{dc + cb}{b + c} \text{ make } \frac{c - d}{b + c} = 2x.$$

And if you please, for $\frac{dc + cb}{b + c}$ put z .

Then will $aa - 2xa = z$ be the new Equation, equal to the other, being now fitted for a Solution.

Now any of these three Forms of Equations being thus prepared for a Solution, may be reduced to simple Powers, by casting off the second or lowest Term of the unknown Quantity; which is done by Substitution, thus, always take half the known Co-efficient, and Add it to (Case 1.) or Subtract it from (Case 2.) its fellow Factor; and for their Sum, or Difference Substitute another Letter. As in these.

Let	1	$aa + 2ba = dc$	Case 1.
Put	2	$a + b = e$	
2 $\odot$ 2	3	$aa + 2ba + bb = ee$	
3 $-$ 1	4	$bb = ee - dc$	
4 $+$ dc	5	$ee = bb + dc$	
5 $\sqrt{\quad}$ 2	6	$e = \sqrt{bb + dc}$	
2 and 6	7	$a + b = \sqrt{bb + dc}$	Per Axiom 5.
7 $-$ b	8	$a = \sqrt{bb + dc} - b$	

Again.

Let	1	$aa - 2ba = dc$	Case 2.
Put	2	$a - b = e$	
2 $\odot$ 2	3	$aa - 2ba + bb = ee$	
3 $-$ 1	4	$bb = ee - dc$	
4 $+$ dc	5	$ee = dc + bb$	
5 $\sqrt{\quad}$ 2	6	$e = \sqrt{dc + bb}$	
2, 6	7	$a - b = \sqrt{dc + bb}$	
7 $+$ b	8	$a = b + \sqrt{dc + bb}$	

In Case 3. From half the known Co-efficient Subtract its fellow Factor.

Thus, Let	1	$2ba - aa = dc$	
Put	2	$b - a = e$	
2 $\odot$ 2	3	$bb - 2ba + aa = ee$	
1 $+$ 3	4	$bb = dc + ee$	
4 $-$ dc	5	$ee = bb - dc$	

5	uw 2	6	$c = \sqrt{bb - dc}$
(2,	6	7	$b - a = \sqrt{bb - dc}$
7 + a	8	$b = a + \sqrt{bb - dc}$	
8 = $\sqrt{\&c.}$	9	$a = b - \sqrt{bb - dc}$	

And this Method holds good in those other Equations, wherein the highest Powers are $a^4, a^6, a^8, \&c.$ As for Instance.

Let	1	$a^6 + 2ba^3 = dc.$	Case 1.
Put	2	$a^3 + b = e$	
2 $\odot$ 2	3	$a^6 + 2ba^3 + bb = ee$	
3 - 1	4	$bb = ee - dc$	
4 + dc	5	$ee = bb + dc$	
3 uw 2	6	$e = \sqrt{bb + dc}$	
2,	6	7	$a^3 + b = \sqrt{bb + dc}$
7 - b	8	$a^3 = \sqrt{bb + dc} - b$	
8 uw 3	9	$a = \sqrt[3]{\sqrt{bb + dc} - b}$	

The same may be done with all the Rest, Care being taken to Add, or Subtract, according as the Case Requires.

But all Quadratick Equations, may be more easily Resolved by Compleating the Square which is grounded upon the Consideration of Raising a Square from any Binominal, or Residual Root. (See Sect. 5. Chap. 1.)

Viz. If $a + b$ be involved to a Square, it will be, $aa + 2ba + bb$

And if $a - b$ be so Involved, it will be, $aa - 2ba + bb$

Whence it is easie to observe, that $aa + 2ba = dc.$ Case 1.

And $aa - 2ba = dc.$ Case 2. Are imperfect Squares, wanting only bb to make them Compleat. And therefore it is that if half the known Co-efficient, be involved to the Second Power, and that Square be Added to both Sides of the Equation, the unknown Side will become a Compleat Square.

Thus, Let	1	$aa + 2ba = dc.$	$\left\{ \begin{array}{l} \text{Here half the C-coefficient} \\ 2b \text{ is } b, \text{ which being} \\ \text{squared, is } bb. \end{array} \right.$
But	2	$bb = bb$	
1 + 2	3	$aa + 2ba + bb = dc + bb$	Case 1.
3 uw 2	4	$a + b = \sqrt{dc + bb} \&c.$	As before.

Again,

Again.

Let	1	$aa - 2ba = dc$	Case 2.
But	2	$bb = bb$	
1 + 2	3	$aa - 2ba + bb = dc + bb$	
3 uw 2	4	$a - b = \sqrt{dc + bb}$	&c. As before.

But, in Case 3. you must Change the Signs of all the Terms in the Equation.

Thus,	1	$2ba - aa = dc$	Case 3.
1 +	2	$aa - 2ba = -dc$	
Then	3	$aa - 2ba + bb = bb - dc$	&c.

And this Method of Compleating the Square holds True in those other Equations.

Viz.	1	$aaaa + 2baa = dc$	Case 1.
For	2	$bb = bb$	As before.
1 + 2	3	$aaaa + 2baa + bb = dc + bb$	
3 uw 2	4	$aa + b = \sqrt{dc + bb}$	
4 - b	5	$aa = \sqrt{dc + bb} : -b$	
5 uw 2	6	$a = \sqrt{\sqrt{dc + bb} : -b}$	And so for the rest.

Or let	1	$a^6 + 2baaa = dc$	As before, Case 1.
And	2	$bb = bb$	
1 + 2	3	$a^6 + 2baaa + bb = dc + bb$	
3 uw 2	4	$aaa + b = \sqrt{dc + bb}$	
4 - b	5	$aaa = \sqrt{dc + bb} : -b$	
5 uw 3	6	$a = \sqrt[3]{\sqrt{dc + bb} : -b}$	&c.

Corollary.

Hence it's evident, that whatsoever Method is used in Solving these (or indeed any other) Equations, the Result will still be the same, if the Work be true; as you may observe from the Operations of this Section: For both these Methods here proposed, give the same Theorems in their respective Cases for the Value of (a.)

Thus

Thus, when $aa + 2ba = dc$ Then

Theorem 1. $a = \sqrt{dc + bb} : -b$

And when $aa - 2ba = dc$ Then

Theorem 2. $a = b + \sqrt{dc + bb}$

Again, when $2ba - aa = dc$ Then

Theorem 3. $a = b - \sqrt{bb - dc}$

The like Theorems may be easily raised for the rest.

If the known Co-efficients (of the second or lowest Term) be any single Quantity, as $aa + ba = dc$, &c. Then is $\frac{1}{2}b$ its Half, and $\frac{1}{4}bb$ will be the Square of that Half: That is, $\frac{1}{2}b \times \frac{1}{2}b = \frac{1}{4}bb$. And then the Work will stand

Thus,	1	$aa + ba = dc$
1 C □	2	$aa + ba + \frac{1}{4}bb = dc + \frac{1}{4}bb$
2 uw 2	3	$a + \frac{1}{2}b = \sqrt{dc + \frac{1}{4}bb}$
3 — $\frac{1}{2}b$	4	$a = \sqrt{dc + \frac{1}{4}bb} : -\frac{1}{2}b$. And so for the rest.

Note, C □ placed in the Margin against the second Step, signifies that the imperfect Square $aa + ba$ in the first Step, is there Completed, viz, in the second Step.

Now by help of these Theorems, it will be easie to Calculate or find the Value of the unknown Quantity (a) in Numbers.

Example 1.

Suppose $aa + 2ba = z$. Let $b = 16$ And $z = 4644$.

Then $a = \sqrt{z + bb} : -b$ Per Theorem 1.

But $z + bb = 4644 + 256 = 4900$ And $\sqrt{4900} = 70$

Consequently $a = 70 - 16$, viz. $a = 54$.

But every Adfected Equation, hath as many Roots (or rather Values of the unknown Quantity) either Real or Imaginary, as are the Dimensions (viz. the Index) of its highest Power; and therefore the Quantity a , in this Equation, hath another Value either Affirmative or Negative; which may be thus found.

The given Equation is $aa + 32a = 4644$, and its Root $a = 54$. Let these two Equations be made or Equated equal to 0, viz. to Nothing.

Thus,

Thus, $aa + 32a - 4644 = 0$. And $a - 54 = 0$.

Then Divide the given Equation by its first Root, and the Quotient will shew the second Value of a .

$$\begin{array}{r} \text{Thus, } a - 54 = 0 \quad aa + 32a - 4644 = 0 \quad (a + 86 = 0) \\ \quad \quad \quad aa - 54a \\ \quad \quad \quad + 86a - 4644 \\ \quad \quad \quad 86a - 4644 \\ \quad \quad \quad (0) \end{array}$$

Hence the second Value of a is $= -86$, Or $86 = -a$ which seems Impossible, viz. that an Affirmative Quantity should be Equal to a Negative Quantity; yet even by this second Value of a , and the same Co-efficient, the true (or first) Equation may be formed.

Thus, Let	1	$a = -86$	
1 $\odot$ 2	2	$aa = +7396$, viz. $-86 \times -86 = +7396$	
1 $\times$ 32	3	$32a = -2752$	
2 $+$ 3	4	$aa + 32a = 4644$	As at First.

Example 2.

Suppose	1	$aa - 7a = 948,75$	Then per Theorem 2.
1 $\odot$ $\square$	2	$aa - 7a + \frac{49}{4} = 948,75 + \frac{49}{4} = 961$	
2 $\sqrt{\quad}$	3	$a - \frac{7}{2}$ (Or 3,5) $= \sqrt{961} = 31$	
3 $+$ 3,5	4	$a = 31 + 3,5 = 34,5$	

Again, for the second Value of a ,

Let $aa - 7a - 948,75 = 0$. And $a - 34,5 = 0$
 Then, $a - 34,5 = 0$ $aa - 7a - 948,75 = 0$ $(a + 27 = 0)$
 Consequently this second Value is $a = -27,5$
 Which will form the Original Equation, $aa - 7a = 948,75$
 if it be ordered as the last was.

Example 3.

Suppose $36a - aa = 243$. Then per Theorem 3.

$$a = 18 - \sqrt{324 - 243} \quad \text{viz. half } 36 \text{ squared is } 324 \text{ \&c.}$$

That is, $a = 18 - \sqrt{81}$ but $\sqrt{81} = 9$

Therefore $a = 18 - 9 = 9$. Now this third Form is called an Ambiguous Equation, because it hath Two Affirmative Values of the unknown Quantity, (a) both which may be found without such Division, as was used before.

For in this Case, $a = 18 \pm \sqrt{81}$, viz. $a = 18 + 9 = 27$,
 Or $a = 18 - 9 = 9$, as before; And both these Values of a
 are equally true, as to forming the given Equation;
 Viz. $36a - aa = 243$. For if $a = 9$, then $aa = 81$,
 and $36a = 324$; but $324 - 81 = 243$, therefore $a = 9$.
 Again, if $a = 27$, then will $aa = 729$ and $36a = 972$.
 But $972 - 729 = 243$, consequently it may be, $a = 27$.

Now either of these Values of a may be found by Division,
 as those were in the other two Cases, one of them being First
 found by the Theorem.

Thus, let $36a - aa - 243 = 0$ And $9 - a = 0$

Then $9 - a = 0$ $36a - aa - 243 = 0$ ($a - 27 = 0$)

$$\begin{array}{r} 9a - aa \\ 27a - 0 - 243 \\ 27a - 243 \\ \hline (0) \quad (0) \end{array}$$

Hence, if $a - 27 = 0$ Then $a = 27$ As before.

Norwithstanding all Quadratick Equations of this third Form
 have Two Affirmative Roots, (as in this) yet but one of those
 Roots will give a true Answer to the Question, and that is to be
 chosen according to the Nature and Limits of the Question, as
 shall be shewed further on.

Scholium.

From the Work of the three last Examples, it may be observed;
 That the Sum of both the Roots, will always be equal to the Co-efficient
 of their respective Equations, with a contrary Sign.

Thus. In Example 1. $aa + 32a = 4644$

Here $a = 54$
 And $a = -86$ } Add

$$2a = -32$$

In Example 2. $aa - 7a = 948,75$

Here $a = 34,5$
 And $a = -27,5$ } Add

$$2a = +7$$

In the last Example $36a - aa = 243$

Which was changed into $aa - 36a = -243$

Here $a = 9$
 And $a = 27$ } Add

$$2a = 36$$

D d

Hence

Hence it's *Evident*, that if either of the *Roots* be found, the other may be *easily* had without *Division*.

If the *Contents* of this *Section* be well understood, it will be *easy* to give a *Numerical Solution* to any *Quadratick Equation*, that happens to arise in *Resolving* of *Questions*, &c. And as for giving a *Geometrical Construction* of them, I think it not *Proper* in this *Place*; because I here suppose the *Learner* wholly *Ignorant* of the first *Principles* of *Geometry*, therefore I shall refer that *Work* to the next *Part*.

CHAP. IX.

Of Analysis, or the Method of Resolving Problems; Exemplified by Variety of Numerical Questions.

N. B. Here I advise the young *Learner*, to make use always of the same *Letters*, to represent the same *Data* in all *Questions*.

Viz. { If *a* represent any Number } or other Quantity.
 { And *e* represent a less Number }

Then let $\left\{ \begin{array}{l} a + e = s \\ a - e = d \\ ae = p \\ \frac{a}{e} = q \end{array} \right\}$ Their $\left\{ \begin{array}{l} \text{Sum.} \\ \text{Difference.} \\ \text{Product.} \\ \text{Quotient.} \end{array} \right\}$

$\left\{ \begin{array}{l} aa + ee = z \\ aa - ee = x \end{array} \right\}$ The $\left\{ \begin{array}{l} \text{Sum of their Squares.} \\ \text{Difference of their Squares.} \end{array} \right\}$

Any Two of these six (*s*, *d*, *p*, *q*, *z*, *x*) being given, thence to find the Rest; which admits of fifteen Variations or Questions.

Question 1. Suppose *s* and *d* were given, and it were required by them to find *a* . *e* . *p* . *q* . *z* . and *x*.

Let $\left\{ \begin{array}{l} 1 \quad a + e = s \\ 2 \quad a - e = d \end{array} \right\}$ and suppose $\left\{ \begin{array}{l} s = 240 \\ d = 192 \end{array} \right\}$ Then

$1 + 2 \quad 3 \quad 2a = s + d = 432$

$3 \div 2 \quad 4 \quad a = \frac{432}{2} = 216$ Here *a* is found.

$1 - 2 \quad 5 \quad 2e = s - d = 48$

$5 \div 2$	6	$e = \frac{s+d}{2} = 24$	Here e is found.
4×6	7	$ae = \frac{ss-dd}{4} = p = 5184$	Here p is found.
$4 \div 6$	8	$\frac{a}{e} = \frac{s+d}{s-d} = q = 9$	Here q is found.
$4 \odot 2$	9	$aa = \frac{ss + 2sd + dd}{4} = 46656$	
$6 \odot 2$	10	$ee = \frac{ss - 2sd + dd}{4} = 576$	
$9 + 10$	11	$aa + ee = \frac{ss + dd}{2} = z = 47232$	z found.
$9 - 10$	12	$aa - ee = sd = x = 46080$	x found.

Question 2. Let s and p be given; To find the Rest.

That is, {	1	$a + e = s = 240$	} Quere a, e, d, q, z, x .
	2	$ae = p = 5184$	
$1 \odot 2$	3	$aa + 2ae + ee = ss = 57600$	
2×4	4	$4ae = 4p = 20736$	
$3 - 4$	5	$aa - 2ae + ee = ss - 4p = 36864$	
$5 \text{ uw } 2$	6	$a - e = \sqrt{ss - 4p} = d = 192$	
$1 + 6$	7	$2a = s + \sqrt{ss - 4p}$	
$7 \div 2$	8	$a = \frac{s + \sqrt{ss - 4p}}{2}$	Here a is found = 216
$1 - 6$	9	$2e = s - \sqrt{ss - 4p}$	
$9 \div 2$	10	$e = \frac{s - \sqrt{ss - 4p}}{2}$	Here e is found = 24.
$8 \div 10$	11	$\frac{a}{e} = \frac{s + \sqrt{ss - 4p}}{s - \sqrt{ss - 4p}} = q = 9$	
$8 \odot 2$	12	$aa = \frac{ss + s\sqrt{ss - 4p} : - p}{2}$	
$10 \odot 2$	13	$ee = \frac{ss - s\sqrt{ss - 4p} : - p}{2}$	

$$12 + 13 \mid 14 \mid aa + ee = ss - 2p = z = 47232$$

$$12 - 13 \mid 15 \mid aa - ee = s\sqrt{ss - 4p} = x = 46080$$

Question 3. Suppose s and q are given, To find the Rest.

$$\text{Viz. } \left\{ \begin{array}{l} 1 \mid a + e = s = 240 \\ 2 \mid \frac{a}{e} = q = 9 \end{array} \right\} \text{ Quere } a . e . d . p . z . x$$

$$2 \times e \mid 3 \mid a = qe$$

$$1 - 3 \mid 4 \mid e = s - qe$$

$$4 + qe \mid 5 \mid qe + e = s$$

$$5 \div q + 1 \mid 6 \mid e = \frac{s}{q + 1} \quad \text{For } q + 1 \times e = qe + e$$

$$1 - 6 \mid 7 \mid a = s - \frac{s}{q + 1} = \frac{qs}{q + 1}$$

$$6 \times 7 \mid 8 \mid ae = \frac{qss}{qq + 2q + 1} = p$$

$$7 - 6 \mid 9 \mid a - e = \frac{qs - s}{q + 1} = d$$

$$7 \odot 2 \mid 10 \mid aa = \frac{qqss}{qq + 2q + 1}$$

$$6 \odot 2 \mid 11 \mid ee = \frac{ss}{qq + 2q + 1}$$

$$10 + 11 \mid 12 \mid aa + ee = \frac{qqss + ss}{qq + 2q + 1} = z$$

$$10 - 11 \mid 13 \mid aa - ee = \frac{qqss - ss}{qq + 2q + 1} = x$$

Question 4. Let s and z be given; To find the Rest.

$$\text{Viz. } \left\{ \begin{array}{l} 1 \mid a + e = s = 240 \\ 2 \mid aa + ee = z = 47232 \end{array} \right\} \text{ Quere } a . e . d . p . q . x$$

$$1 \odot 2 \mid 3 \mid aa + 2ae + ee = ss$$

$$3 - 2 \mid 4 \mid 2ae = ss - z$$

$$2 - 4 \mid 5 \mid aa - 2ae + ee = 2z - ss$$

$$5 \div 2 \mid 6 \mid a - e = \sqrt{2z - ss} = d$$

$1 + 6$	7	$2a = s + \sqrt{2z - ss}$
$7 \div 2$	8	$a = \frac{s + \sqrt{2z - ss}}{2}$
$1 - 6$	9	$2e = s - \sqrt{2z - ss}$
$9 \div 2$	10	$e = \frac{s - \sqrt{2z - ss}}{2}$

The Rest are found
just as in the 2d Que-
stion; the 8 and 10
Steps here, being the
very same with the
8 and 10 Steps there.

Question 5. When s and x are given; To find the Rest.

Viz. { $\begin{array}{l} 1 \quad a + e = s = 240 \\ 2 \quad aa - ee = x = 46080 \end{array} \}$ Quere $a. e. d. p. q. z.$

$2 \div 1$	3	$a - e = \frac{x}{s} = d$ viz. $a + e) aa - ee (a - e$
$1 + 3$	4	$2a = s + \frac{x}{s} = \frac{ss + x}{s}$
$4 \div 2$	5	$a = \frac{ss + x}{2s}$
$1 - 3$	6	$2e = s - \frac{x}{s} = \frac{ss - x}{s}$
$6 \div 2$	7	$e = \frac{ss - x}{2s}$
5×7	8	$ae = \frac{ssss - xx}{4ss} = p$
$5 \div 7$	9	$\frac{a}{e} = \frac{ss + x}{ss - x} = q$
$5 \odot 2$	10	$aa = \frac{s^4 + 2ssx + xx}{4ss}$
$7 \odot 2$	11	$ee = \frac{s^4 - 2ssx + xx}{4ss}$
$10 \div 11$	12	$aa + ee = \frac{s^4 + xx}{2ss} = z$

Question 6. Suppose d and p are given; To find the Rest.

Viz. { $\begin{array}{l} 1 \quad a - e = d = 192 \\ 2 \quad ae = p = 5184 \end{array} \}$ Quere $a. e. s. q. z. x.$

$1 \odot 2$

1 ⊖ 2	3	$aa - 2ae + ee = dd$
2 × 4	4	$4ae = 4p$
3 + 4	5	$aa + 2ae + ee = dd + 4p$
5 √ 2	6	$a + e = \sqrt{dd + 4p} = s$
6 + 1	7	$2a = d + \sqrt{dd + 4p}$
7 ÷ 2	8	$a = \frac{d + \sqrt{dd + 4p}}{2}$
6 - 1	9	$2e = \sqrt{dd + 4p} - d$
9 ÷ 2	10	$e = \frac{\sqrt{dd + 4p} - d}{2}$
8 ÷ 10	11	$\frac{a}{e} = \frac{d + \sqrt{dd + 4p}}{\sqrt{dd + 4p} - d} = q$
8 ⊖ 2	12	$aa = \frac{dd + 2p + d\sqrt{dd + 4p}}{2}$
10 ⊖ 2	13	$ee = \frac{dd + 2p - d\sqrt{dd + 4p}}{2}$
12 + 13	14	$aa + ee = dd + 2p = r$
12 - 13	15	$aa - ee = d\sqrt{dd + 4p} = x$

Question 7. Let d and q be given; To find the Rest.

Viz. $\left\{ \begin{array}{l} 1 \ a - e = d = 192 \\ 2 \ \frac{a}{e} = q = 9 \end{array} \right\}$ Quere a, e, s, p, r, x .

2 × e	3	$a = qe$
1 + e	4	$a = d + e$
3, 4	5	$qe = d + e$
5 - e	6	$qe - e = d$
6 ÷ q - 1	7	$e = \frac{d}{q - 1}$ For $q - 1 \times e = qe - e$
1 + 7	8	$a = d + \frac{d}{q - 1} = \frac{qd}{q - 1}$
7 + 8	9	$a + e = \frac{qd + d}{q - 1} = s$

7 x 8	10	$ae = \frac{qdd}{qq - 2q + 1} = p$
8 ⊙ 2	11	$aa = \frac{qqdd}{qq - 2q + 1}$
7 ⊙ 2	12	$ee = \frac{dd}{qq - 2q + 1}$
11 + 12	13	$aa + ee = \frac{qqdd + dd}{qq - 2q + 1} = z$
11 - 12	14	$aa - ee = \frac{qqdd - dd}{qq - 2q + 1} = x$

Question 8. Suppose d and z given; To find the Rest.

Viz. { $\begin{array}{l} 1 \quad a - e = d = 192 \\ 2 \quad aa + ee = z = 47232 \end{array} \right\}$ Quere a, e, s, p, q, x .

1 ⊙ 2 3 $aa - 2ae + ee = dd$

2 - 3 4 $2ae = z - dd$

2 + 4 5 $aa + 2ae + ee = 2z - dd$

5 w 2 6 $a + e = \sqrt{2z - dd} = s$

1 + 6 7 $2a = d + \sqrt{2z - dd}$

7 ÷ 2 8 $a = \frac{d + \sqrt{2z - dd}}{2}$

6 - 1 9 $2e = \sqrt{2z - dd} - d$

9 ÷ 2 10 $e = \frac{\sqrt{2z - dd} - d}{2}$

8 x 10 11 $ae = \frac{z - dd}{2} = p$

8 ⊙ 2 12 $aa = \frac{z + d\sqrt{2z - dd}}{2}$

10 ⊙ 2 13 $ee = \frac{z - d\sqrt{2z - dd}}{2}$

12 - 13 14 $aa - ee = d\sqrt{2z - dd} = x$

8 ÷ 10 15 $\frac{a}{e} = \frac{d + \sqrt{2z - dd}}{\sqrt{2z - dd} - d} = q$

Question

Question 9. Let d and x be given; To find the Rest.

Viz. $\left\{ \begin{array}{l} 1 \quad a - e = d = 240 \\ 2 \quad aa - ee = x = 46080 \end{array} \right\}$ Quere $a. e. s. p. q. r.$

$$2 \div 1 \quad 3 \quad a + e = \frac{x}{d} = s \text{ viz. } a - e) aa - ee (a + e$$

$$1 + 3 \quad 4 \quad 2a = \frac{dd + x}{d}$$

$$4 \div 2 \quad 5 \quad a = \frac{dd + x}{2d}$$

$$3 - 5 \quad 6 \quad e = \frac{x - dd}{2d}$$

$$5 \times 6 \quad 7 \quad ae = \frac{xx - d^4}{4dd} = p$$

$$5 \div 6 \quad 8 \quad \frac{a}{e} = \frac{dd + x}{x - dd} = q$$

$$5 \odot 2 \quad 9 \quad aa = \frac{d^4 + 2ddx + xx}{4dd}$$

$$6 \odot 2 \quad 10 \quad ee = \frac{xx - 2ddx + d^4}{4dd}$$

$$9 + 10 \quad 11 \quad aa + ee = \frac{d^4 + xx}{2dd} = r$$

Question 10. Let p and q be given; To find the Rest.

Viz. $\left\{ \begin{array}{l} 1 \quad ae = p = 5184 \\ 2 \quad \frac{a}{e} = q = 9 \end{array} \right\}$ Quere $a. e. d. r. x.$

$$1 \times 2 \quad 3 \quad aa = qp \quad \text{For } \frac{ae}{1} \times \frac{a}{e} = \frac{aae}{e} = aa$$

$$3 \div 2 \quad 4 \quad a = \sqrt{qp}$$

$$1 \div 2 \quad 5 \quad ee = \frac{p}{q} \quad \text{For } \frac{a}{e} \div \frac{ae}{1} = \frac{a}{e} \div \frac{ae}{a} = ee$$

$$5 \div 2 \quad 6 \quad e = \sqrt{\frac{p}{q}}$$

$$4 + 6 \quad 7 \quad a + e = \sqrt{qp} + \sqrt{\frac{p}{q}} = s$$

4 - 6	8	$a - e = \sqrt{qp} - \sqrt{\frac{p}{q}} = d$
3 + 5	9	$aa + ee = qp + \frac{p}{q} = z$
3 - 5	10	$aa - ee = qp - \frac{p}{q} = x$

Question 11. Let p and z be given; To find the Rest.

Viz. {	1	$ae = p = 5184$	} Quere $a, e, \&c.$
	2	$aa + ee = z = 47232$	
1 x 2	3	$2ae = 2p$	
2 + 3	4	$aa + 2ae + ee = z + 2p$	
4 w 2	5	$a + e = \sqrt{z + 2p} = s$	
2 - 3	6	$aa - 2ae + ee = z - 2p$	
6 w 2	7	$a - e = \sqrt{z - 2p} = d$	
5 + 7	8	$2a = \sqrt{z + 2p} + \sqrt{z - 2p}$	
8 ÷ 2	9	$a = \frac{\sqrt{z + 2p} + \sqrt{z - 2p}}{2}$	
5 - 7	10	$2e = \sqrt{z + 2p} - \sqrt{z - 2p}$	
10 ÷ 2	11	$e = \frac{\sqrt{z + 2p} - \sqrt{z - 2p}}{2}$	
9 ÷ 11	12	$\frac{a}{e} = \frac{\sqrt{z + 2p} + \sqrt{z - 2p}}{\sqrt{z + 2p} - \sqrt{z - 2p}} = q$	
9 ⊙ 2	13	$aa = \frac{z + \sqrt{zz - 4pp}}{2}$	
11 ⊙ 2	14	$ee = \frac{z - \sqrt{zz - 4pp}}{2}$	
aa - ee	15	$aa - ee = \sqrt{zz - 4pp} = x$	

Question 12. Let p and x be given; To find the Rest.

Viz {	1	$ae = p = 5184$	} Quere $a, e, \&c.$
	2	$aa - ee = x = 46080$	
1 ⊙ 2	3	$aaee = pp$	

Ee

2 ⊙ 2

2	⊙	2	4	$aaaa - 2aace + eeee = xx$
3	×	4	5	$4aace = 4pp$
4	+	5	6	$aaaa + 2aace + eeee = xx + 4pp$
6	$\sqrt{}$	2	7	$aa + ee = \sqrt{xx + 4pp} = z$
2	+	7	8	$2aa = x + \sqrt{xx + 4pp}$
8	÷	2	9	$aa = \frac{x + \sqrt{xx + 4pp}}{2}$
9	$\sqrt{}$	2	10	$a = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}}$
7	-	2	11	$2ee = \sqrt{xx + 4pp} - x$
11	÷	2	12	$ee = \frac{\sqrt{xx + 4pp} - x}{2}$
12	$\sqrt{}$	2	13	$e = \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}}$
10	+	13	14	$a + e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} + \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}}$
10	-	13	15	$a - e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} - \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}}$
9	+	12	16	$aa + ee = \sqrt{xx + 4pp} = z$

Question 13. Having q and z given; To find the Res.

$\text{Viz. } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right.$	$\frac{a}{e} = q = 9$	} Quere $a, e, \&c$
$\left. \begin{array}{l} 3 \\ 4 \end{array} \right\}$	$aa + ee = z = 47232$	
1 × e	3 $a = qe$	
3 ⊙ 2	4 $aa = qqee$	
2 - 4	5 $ee = z - qqee$	
4 + qqee	6 $qqee + ee = z$	
6 ÷ qq + 1	7 $ee = \frac{z}{qq + 1}$	For $qq + 1 \times ee = qqee + ee$

2 - 7	8	$aq = z - \frac{z}{qq + 1} = \frac{qqz}{qq + 1}$
8 uw 2	9	$a = \sqrt{\frac{qqz}{qq + 1}}$
7 uw 2	10	$e = \sqrt{\frac{z}{qq + 1}}$
9 + 10	11	$a + e = \sqrt{\frac{qqz}{qq + 1}} + \sqrt{\frac{z}{qq + 1}} = s$
9 - 10	12	$a - e = \sqrt{\frac{qqz}{qq + 1}} - \sqrt{\frac{z}{qq + 1}} = d$
9 x 10	13	$ae = \sqrt{\frac{qqzz}{q^4 + 2qq + 1}} = p$
8 - 7	14	$aa - ee = \frac{qqz - z}{qq + 1} = x$

Question 14. When q and x are given; To find the Rest.

Vir. {	1	$\frac{a}{e} = q = 9$	} Quere $a, e, \&c.$
	2	$aa - ee = x = 46080$	
1 x e	3	$a = qe$	
3 @ 2	4	$aa = qgee$	
2 + ee	5	$aa = x + ee$	
4, 5	6	$qgee = x + ee$	
6 - ee	7	$qgee - ee = x$	
7 : qq - 1	8	$ee = \frac{x}{qq - 1}$	
2 + 8	9	$aa = x + \frac{x}{qq - 1} = \frac{qqx}{qq - 1}$	
9 uw 2	10	$a = \sqrt{\frac{qqx}{qq - 1}}$	
8 uw 2	11	$e = \sqrt{\frac{x}{qq - 1}}$	
11 + 12	12	$a + e = \sqrt{\frac{qqx}{qq - 1}} + \sqrt{\frac{x}{qq - 1}} = s$	

10 - 11	13	$a - e = \sqrt{\frac{qqx}{qq - 1}} - \sqrt{\frac{x}{qq - 1}} = d$
10 x 11	14	$ae = \sqrt{\frac{qqxx}{qqq - 2qq + 1}} = p$
8 + 9	15	$aa + ee = \frac{qqx + x}{qq - 1} = z$

Question 15. When z and x are given; To find the Rest.

$Vi\ z.$	{	1	$aa + ee = z = 47232$	} Quere $a, e, \&c.$
		2	$aa - ee = x = 46080$	
1 + 2		3	$2aa = z + x$	
3 $\div$ 2		4	$aa = \frac{z + x}{2}$	
1 - 2		5	$2ee = z - x$	
3 $\div$ 2		6	$ee = \frac{z - x}{2}$	
4 $\sqrt{\quad}$		7	$a = \sqrt{\frac{z + x}{2}}$	
6 $\sqrt{\quad}$		8	$e = \sqrt{\frac{z - x}{2}}$	
7 + 8		9	$a + e = \sqrt{\frac{z + x}{2}} + \sqrt{\frac{z - x}{2}} = s$	
7 - 8		10	$a - e = \sqrt{\frac{z + x}{2}} - \sqrt{\frac{z - x}{2}} = d$	
7 x 8		11	$ae = \sqrt{\frac{zz - xx}{4}} = p$	
7 $\div$ 8		12	$\frac{a}{e} = \frac{\sqrt{z + x}}{\sqrt{z - x}} = q$	

These Fifteen Questions are proposed in Doctor Pell's Algebra but he pursues only the first Question throughout, and breaks off in the other Fourteen, after the Values of what I call a and e are found. But I have proceeded in every One of them

to find the *Values* of all the *unknown Quantities*, because they *Afford* such *Variety*, as being well *observed* by a *Learner*, will be found very useful in the *Solution* of most *Questions*.

Note, I have *chose* to use the same *Numbers* for the *respective Value* of each *Quantity* throughout all the *Questions*, because they will be more *Satisfactory* in *proving* the *Work*, than *various Numbers* would have been. Not but that any *Numbers* may be taken at *Pleasure*, provided that the *Number* represented by *a*, be greater than that by *e*, &c. I have omitted the *Numerical Calculation* purely for the *Learner* to *practise* on.

Question 16. *There are two Numbers, the Sum of their Squares is 2368; And the Greater of them is in Proportion to the Less, As 6 To 1. What are those Numbers?*

Let *a* = the Greater Number, *e* = the Lesser, and $\Sigma = 2368$

Then	1	$aa + ee = \Sigma$	} By the Question,
And	2	$a : e :: 6 : 1$	
2	3	$1a = 6e$	
3	4	$aa = 36ee$	
1	5	$ee = \Sigma - 36ee$	
5	6	$37ee = \Sigma$	
6	7	$ee = \frac{\Sigma}{37} = 64$	
7	8	$e = \sqrt{\frac{\Sigma}{37}} = 8$	Proof {
8	9	$6e = 6\sqrt{\frac{\Sigma}{37}} = 48$	
3,	9	$a = 48$	
	10		

If $a = 48$
 And $e = 8$
 $aa = 2304$
 $ee = 64$
 $aa + ee = 2368$
 And $48 : 8 :: 6 : 1$

Question 17. *There are three Numbers in continued Proportion, the Sum of the Extreames is 156, and the Mean is 72: What are the two Extreames.*

That is, Suppose *a . m . e*. in $::$ and $m = 72$.

Then {	1	$a + e = s = 156$	} By the Question.
	2	$a : m :: m : e$	
2	3	$ae = mm$	} Quere <i>a . e</i> . &c.
1	4	$aa + 2ae + ee = ss$	
3	5	$4ae = 4mm$	

4 — 5	6	$aa - 2ae + ee = ss - 4mm$	
6 $\times$ 2	7	$a - e = \sqrt{ss - 4mm}$	
1 + 7	8	$2a = s + \sqrt{ss - 4mm}$	
8 $\div$ 2	9	$a = \frac{s + \sqrt{ss - 4mm}}{2} = 108$	} Or { $a = 48$ $e = 108$
1 — 9	10	$e = \frac{s - \sqrt{ss - 4mm}}{2} = 48$	

Question 18. There are three Numbers in $\therefore$ their Sum is 74, and the Sum of their Squares is 1924; What are the Numbers?

That is, a, e, y are in $\therefore$

Then {	1	$a + e + y = s = 74$	} Quere a, e, y
	2	$aa + ee + yy = \Sigma = 1924$	
	3	$a : e :: e : y$	
3 $\therefore$	4	$ay = ee$	
1 — e	5	$a + y = s - e$	
2 — ee	6	$aa + yy = \Sigma - ee$	
4 $\times$ 2	7	$2ay = 2ee$	
6 + 7	8	$aa + 2ay + yy = \Sigma + ee$	
5 $\odot$ 2	9	$aa + 2ay + yy = ss - 2se + ee$	
8 and 9	10	$\Sigma + ee = ss - 2se + ee$	
10 —	11	$2se = ss - \Sigma$	
11 $\div$ 2s	12	$e = \frac{ss - \Sigma}{2s} = 24$	
5,	13	$a + y = s - e = 50$	
13 $\odot$ 2	14	$aa + 2y + yy = 2500$	
4 $\times$ 4	15	$4ay = 4ee = 2304$	
14 — 15	16	$aa - 2ay + yy = 196$	
16 $\times$ 2	17	$a - y = \sqrt{196} = 14$	
13 + 17	18	$2a = 50 + 14 = 64$	
18 $\div$ 2	19	$a = 32$	} Or { $a = 18$ $y = 32$
13 — 19	20	$y = 50 - 32 = 18$	

Note. In all Questions about Continual Proportionals, (either Arithmetical or Geometrical) where Three Terms are sought, the Mean is Easiest found First (as above;) and if all the Terms be Affirmative, then 'tis equal whether the First, or Last Term be the Greatest.

Question

Question 19. There are three Numbers in $\div$ their Sum is 76; And if the Sum of the Extreames be Multiplied into the Mean, that Product will be 1248: What are those Numbers?

<i>Viz.</i> {	1	$a : e :: e : y$	} by the Question:
	2	$a + e + y = s = 76$	
	3	$ae + ye = p = 1248$	
1 $\therefore$	4	$ay = ee$	
2 $\times e$	5	$ae + ee + ye = se$	
5 $- 3$	6	$ee = se - p$	
6 $- se$	7	$ee - se = -p$	
7 $C \square$	8	$ee - se + \frac{1}{4}ss = \frac{1}{4}ss - p$	
8 $uw 2$	9	$e - \frac{1}{2}s = \sqrt{\frac{1}{4}ss - p}$	
9 $+ \frac{1}{2}s$	10	$e = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - p} = \begin{cases} 52. & \text{Per Theor. 3.} \\ 24. & \text{Chap. 8.} \end{cases}$	
1 $- 10$	11	$a + y = 52$	
4 $\times 4$	12	$4ay = 4ee = 2304$	
11 $\odot 2$	13	$aa + 2ay + yy = 2704$	
13 $- 12$	14	$aa - 2ay + yy = 400$	
14 $uw 2$	15	$a - y = \sqrt{400} = 20$	
11 $+ 15$	16	$2a = 52 + 20 = 72$	
16 $\therefore 2$	17	$a = 36$	} { Or $a = 16$ and $y = 36$
11 $- 17$	18	$y = 52 - 36 = 16$	

N. B. If you take $e = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - p} = 52$ (at the 10th Step) Then it will be $76 - 52 = 24 = a + y$, which is impossible, viz. that the Mean should be Greater than the Sum of the two Extreames.

Therefore it must be $e = \frac{1}{2}s - \sqrt{\frac{1}{4}ss - p} = 24$. See Page 7.

Question 20. There are three Numbers in Arithmetical Progression, the First being Added to twice the Second, and three times the Third, their Sum will be 62 And the Sum of all their Squares is 275: What are those Numbers?

Suppose	1	a, e, y in Arithmetical Progression.	
And {	2	$a + 2e + 3y = 62$	} by the Question.
	3	$aa + ee + yy = 275$	
Then	4	$a + y = 2e$	Per Sect. 1. Chap. 6.
2 $- 4$	5	$2e - 2y = 62 - 2e$	
5 $\therefore 2$	6	$e + y = 31 - e$	
6 $- e$	7	$y = 31 - 2e$	
4 $- 7$	8	$a = 4e - 31$	

8	⊙	2	9	$aa = 16ee - 248e + 961$
7	⊙	2	10	$yy = 961 - 124e + 4ee$
9	+	10	11	$aa + yy = 20ee - 372e + 1922$
3	—	11	12	$ee = 372e - 20ee - 1647$
12	+	20ee	13	$21ee = 372e - 1647$
13	—	372e	14	$21ee - 372e = -1647$
14	÷	21	15	$ee = \frac{134}{7}e = -\frac{149}{7}$
15	C	□	16	$ee = \frac{134}{7}e + \frac{1844}{49} = \frac{1844}{49} - \frac{149}{7} = \frac{1}{49}$
17	uv	2	17	$e = \frac{62}{7} = \sqrt{\frac{1}{49}} = \frac{1}{7}$ The Mean.
17	+	$\frac{62}{7}$	18	$e = \frac{62}{7} \pm \frac{1}{7} = 9$ Or $8\frac{5}{7}$
18	×	4	19	$4e = 36$ Or $8\frac{6}{7}$
8,	19		20	$a = 36 - 31 = 5$ Or $34\frac{6}{7} - 31 = 3\frac{1}{7}$
18	×	2	21	$2e = 18$ Or $17\frac{1}{7}$
7,	21		22	$y = 31 - 18 = 13$ Or $31 - 17\frac{1}{7} = 13\frac{6}{7}$

Question 21. *There are three Numbers in Arithmetical Progression; The Square of the First Term being added to the Product of the other Two is 576; the Square of the Mean being added to the Product of the two Extreames, makes 612; And the Square of the last Term being added to the Product of the First into the Second, is 792: What are those Numbers?*

Suppose	1	a, e, y	In Arith. Progress. As before.
Then {	2	$aa + ye = 576$	By the Question.
	3	$ee + ya = 612$	
	4	$yy + ae = 792$	
1	∴	5	$a + y = 2e$ Per Sect. 1 Chap. 6.
5	×	6	$ae + ye = 2ee$
2	+	7	$aa + ye + yy + ae = 1368$
7	—	8	$aa + yy = 1368 - 2ee$
3	—	9	$ya = 612 - ee$
9	×	10	$2ya = 1224 - 2ee$
8	+	11	$aa + 2ya + yy = 2592 - 4ee$
5	⊙	12	$aa + 2ya + yy = 4ee$
11,	12	13	$4ee = 2592 - 4ee$
13	+	14	$8ee = 2592$
14	÷	15	$ee = 324$
15	uv	16	$e = \sqrt{324} = 18$ The Mean.
8,	17	$aa + yy = 1368 - 2ee = 720$	
10,	18	$2ya = 1224 - 2ee = 576$	
17	—	19	$aa - 2ya + yy = 720 - 576 = 144$

1	uv	2	20	$a - y = \sqrt{144} = 12$	
5	+	20	21	$2a = 2c - 12 = 48$	
21	$\div$	2	22	$a = 24$	Or $\begin{cases} x = 12 \\ y = 24 \end{cases}$
5	-	22	23	$y = 2c - 24 = 12$	

Question 22. 'Tis required to find two such Numbers, that the Sum of their Squares may be $8226\frac{1}{2}$; And their Product being Added to the Square of the Lesser, may be $6921\frac{1}{2}$.

Viz. {	1	$aa + ee = 8226\frac{1}{2}$	Quere a and e
	2	$ae + ee = 6921\frac{1}{2}$	
1 - 2	3	$aa - ae = 1305$	
3 +	4	$ae = aa - 1305$	
4 $\div$ a	5	$e = \frac{aa - 1305}{a}$	
5 $\odot$ 2	6	$ee = \frac{a^4 - 2610aa - 1703025}{aa}$	
1 - aa	7	$ee = 8226,5 - aa$	
6, 7	8	$\frac{a^4 - 2610aa - 1703025}{aa} = 8226,5 - aa$	
8 $\times$ aa	9	$a^4 - 2610aa - 1703025 = 8226,5 - a^4$	
9 + a ⁴	10	$2a^4 - 2610aa + 1703025 = 8226,5aa$	
10 +	11	$2a^4 - 10836,5aa = -1703025$	
11 $\div$ 2	12	$a^4 - 5418,25aa = -851512,5$ (765	
13 C□	13	$a^4 - 5418,25aa + 7339358,26562 = 6487845,$	
13 uv 2	14	$aa - 2709,125 = \sqrt{6487845,765625} = 2547,$	
14 + 27 &c.	15	$aa = 2709,125 + 2547,125$ (125	
Suppose	16	$aa = 2709,125 + 2547,125 = 5256,25$	
Then	17	$a = \sqrt{5256,25} = 72,5$	
And 5,	18	$e = \frac{aa - 1305}{a} = \frac{5256,25 - 1305}{72,5} = 54,5$	
Or let	19	$aa = 2709,125 - 2547,125 = 162$	
10 uv 2	20	$a = \sqrt{162} = 12,72$ &c.	
Then	21	$e = \frac{162 - 1305}{12,72}$ Which is impossible.	
Therefore		$a = 72,5$	} As at the 17 and 18th Steps.
And		$e = 54,5$	

This Question may be perform'd with less Trouble, by Substituting Letters for the known Numbers.

Viz. $\begin{cases} aa + ee = z \\ ae + ee = p \end{cases}$ Then let $z - p = d = aa - ae$, &c.

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Question 23. It is required to find Three such Numbers, that the Sum of the First and Second, being Multiplied with the Third, may be 37824; And the Sum of the Second and Third, Multiplied with the First, may be 59944; Also, that the Sum of the First and Third, being Multiplied with the Second, may be 52456.

Let a, e, y represent the three Numbers.

Then	{	1	$ay + ey = 37824 = b$	} Quere a, e, y
		2	$ea + ya = 59944 = c$	
		3	$ae + ye = 52456 = d$	
1 + 2 + 3	{	4	$2ae + 2ay + 2ye = b + c + d$	
Let		5	$z = b + c + d$	
4 ÷ 2	{	6	$ae + ay + ye = \frac{1}{2}z = \frac{b + c + d}{2}$	
6 - 3		7	$ay = \frac{1}{2}z - d$	
7 ÷ a	{	8	$y = \frac{z - 2d}{2a}$	
6 - 2		9	$ye = \frac{1}{2}z - c$	
6 - 1	{	10	$ae = \frac{1}{2}z - b = \frac{z - 2b}{2}$	
10 ÷ a		11	$e = \frac{z - 2b}{2a}$	
8 × 11	{	12	$ye = \frac{z - 2d}{2a} \times \frac{z - 2b}{2a} = \frac{zz - 2dz - 2bz + 4bd}{4aa}$	
9, 12		13	$\frac{z - 2c}{2} = \frac{zz - 2dz - 2bz + 4bd}{4aa}$	
13 × 4aa	{	14	$2z^2aa - 4caa = zz - 2dz - 2bz + 4bd$	
14 ÷		15	$aa = \frac{zz - 2dz - 2bz + 4bd}{2z - 4c} = 55696$	
15 √ 2	{	16	$a = \sqrt{55696} = 236$	
11,		17	$e = \frac{z - 2b}{2a} = 158$	
8,	{	18	$y = \frac{z - 2d}{2a} = 96$	

Question 24. 'Tis required to find Two such Numbers, that their Sum being subtracted from the Sum of their Squares, may leave 14. And if their Product be Added to their Sum, it may make 14.

Let a and e be put for the Numbers, and let $y = a + e$

Then {

1	$aa + ee - y = 14$	} By the Question:
2	$ae + y = 14$	

1 + y	3	aa + ee = 14 + y
2 - y	4	ae = 14 - y
4 x 2	5	2ae = 28 - 2y
3 + 5	6	aa + 2ae + ee = 42 - y
6 uw 2	7	a + e = $\sqrt{42 - y}$
But	8	a + e = y By Substitution above.
7, 8,	9	y = $\sqrt{42 - y}$
9 @ 2	10	yy = 42 - y
10 + y	11	yy + y = 42
11 C□	12	yy + y + $\frac{1}{4}$ = 42 + $\frac{1}{4}$ = 42,25
12 uw 2	13	y + $\frac{1}{2}$ = $\sqrt{42,25}$ = 6,5
13 - $\frac{1}{2}$	14	y = 6,5 - $\frac{1}{2}$ = 6
Consequent.	15	a + e = 6 By Restitution from above.
3, 14	16	aa + ee = 14 + 6 = 20
5, 15	17	2ae = 28 - 12 = 16
16 - 17	18	aa - 2ae + ee = 4
18 uw 2	19	a - e = $\sqrt{4}$ = 2
15 + 19	20	2a = 8
23 ÷ 2	21	a = 4 Proof
15 - 21	22	e = 6 - 4 = 2

If a = 4 And e = 2
 Then aa + ee - a - e = 14
 And ae + a + e = 14
 According to the Question.

Question 25. Three Men discoursing of their Money, saith the First, if 100l. were Added to my Money, it would be as much as both your Money put together; said the Second Man, if 100l. were Added to my Money, I should have twice as much as both you have; saith the Third Man, if 100l. were Added to my Money should then have three Times as much Money as both you have; How much Money had each Man?

Let a represent the First Man's Money, e the Second, and y the Third.

Then {	1	a + 100 = e + y	} by the Question.
	2	e + 100 = 2a + 2y	
	3	y + 100 = 3a + 3e	
1 - a	4	e + y - a = 100 = s	} Que. e a, e, y
2 - e	5	2a + 2y - e = 100 = s	
3 - y	6	3a + 3e - y = 100 = s	
4, 6	7	e + y - a = 3a + 3e - y	
7 ±	8	2y = 4a + 2e	
5 - 8	9	2a - e = s - 4a - 2e	
9 + 4a	10	6a + e = s = 100	
4 - 6	11	2a - 4e = 2s = 200	

10×4	12	$24a + 4e = 4s = 400$
$12 - 11$	13	$22a = 2s = 200$
$13 \div 22$	14	$a = \frac{s}{11} = \frac{100}{11} = 9\frac{1}{11}l.$
$10 - 6a$	15	$e = s - 6a = 100 - \frac{600}{11} = \frac{500}{11} = 45\frac{5}{11}l.$
$8 \div 2$	16	$y = 2a + e = \frac{200}{11} + \frac{500}{11} = \frac{700}{11} = 63\frac{7}{11}l.$

Answer. The $\begin{cases} \text{First} \\ \text{Second} \\ \text{Third} \end{cases}$ Man had $\begin{cases} 9l. \ 15s. \ 9\frac{1}{11}d. \\ 45l. \ 9s. \ 1\frac{5}{11}d. \\ 63l. \ 12s. \ 8\frac{7}{11}d. \end{cases}$

Question 26. Three Men have each such a Sum of Money, that if the first and second Mens Money, be Added to half of what the Third Man hath; that Sum will be 92l. And if the second and Third Mens Money, be Added to one Third Part of the first Mans Money, that Sum will be 92l. Lastly, if one fourth part of the second Man's Money, be Added to the first and Third Mens Money, that Sum will also be 92l. How much was each Man's Money.

Put a for the first Man's Money, e for the second, And y for the Third.

Then	$\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right.$	$\begin{cases} a + e + \frac{1}{2}y = s \\ \frac{1}{3}a + e + y = s \\ \frac{1}{4}e + a + y = s \end{cases}$	By the Question. And $s = 92$
$1,$	2	4	$a + e + \frac{1}{2}y = \frac{1}{3}a + e + y$
$4 - e$	5	5	$a + \frac{1}{2}y = \frac{1}{3}a + y$
$5 \times \frac{1}{3}$	6	6	$6a + 3y = 2a + 6y$
$6 +$	7	7	$4a = 3y$
2×3	8	8	$a + 3e + 3y = 3s$
$8 - 7$	9	9	$a + 3e = 3s - 4a$
$9 - a$	10	10	$3e = 3s - 5a$
$10 \div 3$	11	11	$e = \frac{3s - 5a}{3}$
3×4	12	12	$e + 4a + 4y = 4s = 368$
$12 - 2$	13	13	$3\frac{2}{3}a + 3y = 3s = 276$
$13,$	14	14	$3\frac{2}{3}a + 4a = 3s = 276$
14×3	15	15	$11a + 12a = 9s = 828$
$15 \div 23$	16	16	$a = \frac{9s}{23} = \frac{828}{23} = 36l. \text{ The 1st Man's Money.}$
11	17	17	$e = \frac{3s - 5a}{3} = \frac{276 - 180}{3} = 32l. \text{ The 2d Man's Money.}$
$7 \div 3$	18	18	$y = \frac{4a}{3} = \frac{144}{3} = 48l. \text{ The 3d Man's Money.}$

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Question 27. Four Men walking abroad found a Purse of Shillings only, out of which every one took a Number at an Adventure; afterwards by comparing their Numbers together they found; that if the first took 25 Shillings from the second, it would make his Number equal with what the second had then Left. If the second took 30 Shillings from the Third, his Money would then be Triple to what the Third had Left. And if the Third took 40 Shillings from the Fourth, his Money would then be double to what the Fourth had Left. Lastly, The Fourth taking 50 Shillings from the First, he would then have three times as much as the first had Left, and 5 Shillings more.

'Tis required to tell how many Shillings each Man had. Put a for the first Sum, e the second, y the third, and u the Fourth.

Then	}	1	$a + 25 = e - 25$	}	By the Question.
		2	$e + 30 = 3y - 90$		
		3	$y + 40 = 2u - 80$		
		4	$u + 50 = 3a - 145$		
1 + 25		5	$a + 50 = e$		
2 - 30		6	$3y - 120 = e$		
5, 6		7	$a + 50 = 3y - 120$		
7 + 120		8	$a + 170 = 3y$		
8 ÷ 3		9	$y = \frac{a + 170}{3}$		
3 - 40		10	$y = 2u - 120$		
9, 10		11	$2u - 120 = \frac{a + 170}{3}$		
11 + 120		12	$2u = \frac{a + 170}{3} + 120 = \frac{a + 530}{3}$		
12 ÷ 2		13	$u = \frac{a + 530}{6}$		
4 - 50		14	$u = 3a - 195$		
13, 14		15	$3a - 195 = \frac{a + 530}{6}$		
15 × 6		16	$18a - 1170 = a + 530$		
16 ±		17	$17a = 1700$		
17 ÷ 17		18	$a = 100$	The 1st	} Man's N ^o . of Shillings.
by the 5		19	$e = 150$	Second	
by the 9		20	$y = 90$	Third	
by the 14		21	$u = 105$	Fourth	

Question 28. Four Men have each a Sum of Money, which being put all together makes 250 Pounds. And if to the first Man's Money be added 8 Pounds, it will be just as much as the second Man's Money decreased by 8 Pounds, and as much as 8 times the third Man's Money, and but as much as one Eighth part of the fourth Man's Money; How much had each Man?

Let a, e, y, u represent the Four Mens Money.

Then	{	1	$a + e + y + u = s$	{	by the Question.
		2	$a + b = e - b$		Let $s = 250$ and $b = 8$.
		3	$yb = \frac{u}{b} = a + b$		Or any other Number at Pleasure.
2 + b	4	$a + 2b = e$			
3 ÷ b	5	$y = \frac{a + b}{b}$	Because $yb = a + b$		
3 x b	6	$u = ba + bb$	For $\frac{u}{b} = a + b$		
4 + 5 + 6	7	$e + y + u = a + 2b + \frac{a + b}{b} + ba + bb$			
1 - a	8	$e + y + u = s - a$			
7, 8	9	$a + 2b + \frac{a + b}{b} + ba + bb = s - a$			
9 x b	10	$ba + 2bb + a + b + bb + bbb = bs - ba$			
10 ±	11	$2ba + bba + a = bs - bbb - 2bb - b$			
11 ∴	12	$a = \frac{bs - bbb - 2bb - b}{bb + 2b + 1} = 16,691358 \text{ \&c.}$			
by the 4,	13	$e = a + 2b = 32,691358 \text{ \&c.}$			
by the 5,	14	$y = \frac{a + b}{b} = 3,086419 \text{ \&c.}$			
by the 6	15	$u = ba + bb = 197,530864 \text{ \&c.}$			

That is

	l.	s.	d.
$a =$	16	13	9,92592
$e =$	32	13	9,92592
$y =$	3	1	8,74956
$u =$	197	10	7,40736

Consequently $a + e + y + u = 249.19.11,99976$ which should be just 250*l.* the Sum proposed in the Question. Now what it wants of that Sum, proceeds from the Imperfection of the Decimal Parts being not Continued on to more Places, which would have brought it nearer the Truth, tho not perhaps Exactly so. *Vide Sect. 5. Chap. 5. Part 1.*

Ans.

Question 29. Several Merchants enter into Partnership, every one put into the Stock 65 Times as many Pounds as there were Partners; with that Stock they Traded, and gain'd as many Pounds, per 100l. as there were Partners. Now if 10l. 10s. be Added to, and subtracted from their Gain, the Product of that Sum and Difference will be 6491l. 6s. 3d.

Quere, How many Merchants there were, &c.

Let	1	$a =$	The Number of Merchants.
1×65	2	$65a =$	Every one's Sum he put into Stock.
$2 \times a$	3	$65aa =$	The whole Stock.
And	4	$100 : a :: 65aa : \frac{65aaa}{100}$	by the Question.
Viz.	5	$\frac{65aaa}{100} =$	The whole Gain.
$5 + 10,5$	6	$\frac{65aaa}{100} + 10,5$	
$5 - 10,5$	7	$\frac{65aaa}{100} - 10,5$	
6×7	8	$\frac{4225aaaaaa}{10000} - 110,25 = 6491,3125$	by the Quest.
8×10000	9	$4225a^6 - 1102500 = 64913125$	
$9 +$	10	$4225a^6 = 66015625$	
$10 \div 4225$	11	$a^6 = \frac{66015625}{4225} = 15625$	
$11 \text{ w } 6$	12	$a = \sqrt[6]{15625} = 5$	The Number of Merchants
12×65	13	$65a = 325$	The N ^o . of Pounds each put in.

Question. 30. Three Merchants joyn Stocks together; the First Man's Stock was less than the Second Man's by 13l. The Second and Third Man's Stock was 175l. in Trading they gain 48l. more than their whole Stock was; the First Man's proportional Part of the Gain was 78l. What was each Man's Stock, and part of the Gain?

Let a, e, y represent each Man's Stock.

Then {	1	$a + e + y = s$	The whole Stock.
	2	$s + 48 =$	The whole Gain.
And {	3	$a + 13 = e$	By the Question.
	4	$e + y = 175$	
$4 + a$	5	$a + e + y = 175 + a$	
1, 5	6	$s = 175 + a$	

6,	2	7	$s - 48 = 223 + a$	
	But	8	$175 + a : 223 + a :: a : 78$	Per Quest.
8	∴	9	$aa - 223a = 78a + 13650$	
9	$- 78a$	10	$aa - 145a = 13650$	
10	C□	11	$aa - 145a + 5256,25 = 18906,25$	
11	$\sqrt{}$	12	$a + 72,5 = \sqrt{18906,25} = 137,5$	
12	$- 69,5$	13	$a = 137,5 - 72,5 = 65$	
	3,	14	$e = a + 13 = 78$	
4	$- 14$	15	$y = 97$	
	Then	16	$65 : 78 :: 78 : 93l. 12s. = e's$	Gain.
	Again	17	$65 : 78 :: 97 : 116l. 8s. = y's$	Gain.
Proof	{	18	$116l. 8s. + 93l. 12s. + 78l. = 288l.$	The Gain.
		19	$65 + 78 + 97 = 240$	The whole Stock.
18	$- 19$	20	$288 - 240 = 48$	The Gain more than Stock.

Question 31. *A Father at his Death left his Three Son's all his Money in this manner; To the Eldest he gave Half of it, wanting 44 Pounds; To the Second he gave one Third of it, and 14 Pounds more; To the Youngest he gave the Remainder, which was less than the Share of the Second Son by 82 Pounds; What was each Son's Share?*

Let a, e, y be the Three Shares, and $z =$ the whole Sum.

Then	{	1	$a + e + y = z$	
		2	$a = \frac{1}{2}z - 44$	
		3	$e = \frac{1}{3}z + 14$	
		4	$y = \frac{1}{2}z + 14 - 82$	By the Question.
2 + 3 + 4		5	$a + e + y = \frac{2z}{3} + \frac{z}{2} - 98$	
1,	5	6	$z = \frac{2z}{3} + \frac{z}{2} - 98$	
6 x 3		7	$3z = 2z + \frac{3z}{2} - 294$	
7 x 2		8	$6z = 4z + 3z - 588$	
8 ÷		9	$z = 588$	The whole Sum that was left.
2,	9	10	$a = \frac{1}{2}z - 44 = 250$	The El. Son's Share.
3,	9	11	$e = \frac{1}{3}z + 14 = 210$	The Second Son's, &c.
4,	9	12	$y = \frac{1}{2}z + 14 - 82 = 128$	The Youngest, &c.

Question 32. *A Man Playing at Hazard or Dice, won the First Throw just so much Money as he had in his Pocket; the Se-*

cond

Second Throw he won the Square Root of what he then had, and five Shillings more; the Third Throw he won the Square of all he then had; after which his whole Sum was 112l. 16s. What Moneys had he when he began to Play?

Suppose	1	$a =$ His First Sum. Then
1 $\times$ 2	2	$2a =$ His Sum after the First Throw.
And	3	$5 + \sqrt{2a} =$ The Winnings at the second Throw.
2 + 3	4	$2a + 5 + \sqrt{2a} =$ The Sum after the 2d Throw.
4 $\odot$ 2	5	$4aa + 22a + 25 + 4a\sqrt{2a} + 10\sqrt{2a} =$ The Winnings at the Third Throw; And therefore
4 + 5	6	$4aa + 24a + 30 + 4a\sqrt{2a} + 11\sqrt{2a} = 2256$ Shill.

But to avoid these Surd Quantities, let us instead of supposing $a =$ the First Sum, make a Second Trial.

Viz. let	1	$2aa =$ The First Sum.
1 $\times$ 2	2	$4aa =$ The Sum after the First Throw.
Then	3	$2a + 5 =$ The Sum won at the Second Throw.
2 + 3	4	$4aa + 2a + 5 =$ His Sum after the 2d Throw.
4 $\odot$ 2	5	$16a^4 + 16a^3 + 44aa + 20a + 25 =$ The Winnings at the Third Throw; And therefore
4 + 5	6	$16a^4 + 16a^3 + 48aa + 22a + 30 = 2256$ Shil.

Yet again, to avoid these high Equations, let us make a Third Supposition; Thus,

Let	1	$\frac{aa}{2} =$ The First Sum.
1 $\times$ 2	2	$aa =$ The Sum after the First Throw.
Then	3	$a + 5 =$ The Winnings at the 2d Throw.
2 + 3	4	$aa + a + 5 =$ The Sum after the 2d Throw.
Substi.	5	$e = aa + a + 5$
5 $\odot$ 2	6	$ee =$ The Winnings at the 3d Throw. Then
5 + 6	7	$ee + e = 2256$ Shillings by the Question.
7 $\odot$ 2	8	$ee + e + 0,25 = 2256,25$
8 uw 2	9	$e + 0,5 = \sqrt{2256,25} = 47,5$
9 $-$ 0,5	10	$e = 47$
5, 10	11	$aa + a + 5 = 47$
11 $-$ 5	12	$aa + a = 42$
12 $\odot$ 2	13	$aa + a + 0,25 = 42,25$
1 uw 2	14	$a + 0,5 = \sqrt{42,25} = 6,5$
14 $-$ 0,5	15	$a = 6$
15 $\odot$ 2	16	$aa = 36$
16 $\div$ 2	17	$\frac{aa}{2} = \frac{36}{2} = 18$ { The Shil. he had in his Pocket when he began to Play.

Note, In Resolving of this *Question*, I have made Three different Suppositions for the Thing sought; purely as an Instance to shew the Young Learner, how well he ought to consider the Nature of the *Question*, when he first States it, and make choice of representing the Thing sought, So, as to avoid running it into Surds if possible. *viz.* As in the first Supposition of $a =$ the first Sum, &c. Not but that such *Æquations* may be solved, as shall be shewed in the next Chapter. However, it is most like an Artist to perform things of this Nature the Nearest and easiest way they can be done.

Question 33. Suppose there were Two equal Circles, whose Peripheries (*viz.* Circumferences) are divided into 44310 equal Parts; and that those Circles were so placed upon one Axis, as to Move the contrary way to each other; And suppose one of them to Move, but one of those equal Parts the first Day, Two Parts the second Day, Three Parts the third Day, and so on in Arithmetick Progression. *viz.* 1. 2. 3. 4. 5, &c. And the other to Move every Day the Cubes of those Parts, *viz.* 1. 8. 27. 64. 125, &c. of the same Parts. How many Parts, and how many Dayes must each Circle Move, before the same Two Points meet that were together when they began to Move?

In order to give a ready Solution to this *Question* (or any other in this kind) it will be convenient to premise this Lemma.

Lemma.

The Sum of any Series of Cubes, whose Roots are in Arithmetick Progression (the first Term, and common difference being Unity or 1) is equal to the Square of the Sum of all those Roots. As in these

Terms in Arith. &c. Their Cubes

1	1
2	8
3	27
4	64
5	125
6	216
	&c.

Sum $21 \times 21 = 441$ Sum of their Cubes.

Let 1 $a =$ The Sum of all the Parts the 1st Circle moves.
 Then 2 $aa =$ The Sum of all the Parts the 2^d moves.
 Consequen. 3 $aa + a = 44310$ By the Quest. (per Lem.
 3 4 $aa + a + 0,25 = 44310,25$

4 *uw* 3

4	uv	2	5	$a + 0,5 = \sqrt{44310,25} = 210,5$	$\left\{ \begin{array}{l} \text{the Number of Parts the First Circle} \\ \text{must Move.} \end{array} \right.$
5	$-$	$0,5$	6	$a = 210$	
6	$\odot$	2	7	$aa = 44100$	$\left\{ \begin{array}{l} \text{The Number of Parts the Second} \\ \text{Circle Moves.} \end{array} \right.$

Next to find the Number of Days they Moved, There is given, the First Term $= 1$, the common Difference $= 1$, And the Sum of all the Terms $= 210$, thence to find the Last Term, which in this Case is the same with the Number of all the Terms.

Let $a = 1$ the First Term, $e = 1$ the common Difference, And $s = 210$ the Sum of all the Terms, To find $y =$ the Last Term. As per Sect. 1. Chap. 6. Then $yy + ey = 2s + aa - ae$ by the 16th Step Page 186. That is, $yy + y = 210 \times 2 = 420$ &c. Hence $y = 20$ the Number of Days required.

I shall now proceed to give an Example or Two of the Method used in Arguing about Unlimited Questions, viz. such Questions which admit of various Answers, such as those in Alligation Alternate, promised in Page 117.

In order to shorten that Work, it will be convenient for the Learner to know the Two Signs of Comparison $>$ And $<$. The Sign $>$ is of Greater than, As $b > a$ signifies that b is Greater than a . The Sign $<$ is of Lesser than. As $b < d$ signifies that b is Lesser than d . &c.

Example. 1.

Question 33. A Tobacconist hath Three Sorts of Tobacco, viz. one of 2s. 8d. the Pound, another of 20d. the Pound, and a Third Sort of 16d. the Pound; of these he would make a Mixture to contain 56 Pound that may be Sold for 22d. the Pound: How much of each Sort may he take?

Let $a =$ the Quantity of that Worth 32 Pence the Pound, $e =$ that of 20 Pence the Pound, And $y =$ that of 16 Pence the Pound;

Then $a + e + y = 56$
 And $32a + 20e + 16y = 1232$

$\left\{ \begin{array}{l} \text{viz. each Quantity Multi-} \\ \text{plied into its own Price,} \\ \text{Equals their Sum Multi-} \\ \text{plied into the Mean Price.} \end{array} \right.$

This

This *Question* being thus *Stated*, it appears by *Rule 1.* *Page 176.* that it is capable of *Innumerable Answers*; because for any one of these *Three Letters* $a . e . y$. there may be taken any *Number* at *Pleasure*, provided it be *Less* than 56. But altho' that may be truly done, yet there are several ways of *Arguing* about these sort of *Questions*, which will *Limit* or *Bound* them to all their proper or possible *Answers* in whole *Numbers*. Thus.

Let	1	$a + e + y = 56$	} As above.
And	2	$32a + 20e + 16y = 1232$	
1 — a	3	$e + y = 56 - a$	
2 — $32a$	4	$20e + 16y = 1232 - 32a$	
3 × 16	5	$16e + 16y = 896 - 16a$	
4 — 5	6	$4e = 336 - 16a$	
6 ÷ 4	7	$e = 84 - 4a$	Hence $a < \frac{84}{4} = 21$
3 — 7	8	$y = 3a - 28$	Hence $a > \frac{28}{3} = 9\frac{1}{3}$

From the *Two Last Steps* it appears, that the *Quantity* signified by a , ought to be *Less* than 21 , and *Greater* than $9\frac{1}{3}$; That is, any *Number* betwixt $9\frac{1}{3}$ and 21 , may be taken for the *Value* of a . Consequently there may be *Eleven Answers* to this *Question* in *Whole Numbers*.

Suppose $a = 10$ Then $e = 84 - 40 = 44$ Per 7th Step
And $y = 30 - 28 = 2$ Per 8th Step.

Again, if $a = 11$ Then $e = 84 - 44 = 40$ Per 7th Step;
And $y = 33 - 28 = 5$ Per 8th Step. And so on for the *Rest*, which will be as in the following *Table*.

a	e	y	a	e	y	a	e	y
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

Thus it will be *easie* to find out and *Collect* all the *Limited Answers* to any *Question* (of this kind) wherein there are only *Three Quantities* propos'd to be *Mix'd*: But when there are *More* than *Three*, then the *Work* requires a little more *Trouble*; because the *single Limits* of all the *Quantities* above *Two*, must be found. That is, if there are *Four Quantities* concern'd in the *Question*, the *Limits* of *Two* of them must be found; If *Five Quantities* are concern'd, then the *Limits* of *Three*

Three of them must be found, &c. As in the following Question.

Question 34. Suppose it were required to Mix Four Sorts of Wines together; viz. one Sort worth 7s. 4d. the Gallon; another Sort worth 4s. 7d. the Gallon; a Third Sort worth 3s. 8d. the Gallon; And a Fourth Sort worth 2s. 9d. the Gallon; How much of each Sort may be taken to make a Mixture of 63 Gallons, so as that the whole Quantity may be Sold for 5s. 6d. the Gallon; without Loss, &c.

First let all these several Rates, and the Mean Rate, be Reduced to one Denomination, viz. into Pence.

Viz. $\left\{ \begin{array}{l} 7s. 4d = 88d. \\ 3s. 8d = 44d. \end{array} \right. \quad \left\{ \begin{array}{l} 4s. 7d = 55d. \\ 2s. 9d = 33d. \end{array} \right. \text{ And } 5s. 6d = 66d$

Then put a = the Quantity of that Worth 88d the Gallon; e = that of 55d the Gallon; y = that of 44d the Gallon; And u = that of 33d the Gallon.

Then	1	$a + e + y + u = 63$	By the Question.
And	2	$88a + 55e + 44y + 33u = 4158$	$= 63 \times 66$
$- a$	3	$e + y + u = 63 - a$	
$- 88a$	4	$55e + 44y + 33u = 4158 - 88a$	
$\times \frac{33}{33}$	5	$33e + 33y + 33u = 2079 - 33a$	
$- 5$	6	$22e + 11y = 2079 - 55a$	
$\div \frac{11}{11}$	7	$2e + y = 189 - 5a$	Hence $a < \frac{189}{5} = 37\frac{4}{5}$
$\times \frac{55}{55}$	8	$55e + 55y + 55u = 3465 - 55a$	
$- 4$	9	$11y + 22u = 33a - 693$	
$\div \frac{11}{11}$	10	$y + 2u = 3a - 63$	Hence $a > \frac{63}{3} = 21$

From the 7th and 10th Steps it appears, that the Quantity of that Sort of Wine denoted by a , must be Less than $37\frac{4}{5}$ Gallons, and Greater than 21 Gallons: That is, it may be = any Number of Gallons betwixt 21 and $37\frac{4}{5}$. Whence it follows, that there may be Collected 16 Answers to this Question from the Limits of a only.

Next to find the Limits of e , y , and u .

Suppose	11	$a = 22$	Then will $5a = 110$. And $3a = 66$
But	12	$2e + y = 189 - 5a = 79$	Per 7th Step.
$- 2e$	13	$y = 79 - 2e$	Hence $e < \frac{79}{2} = 39\frac{1}{2}$
Again	14	$e + y + u = 63 - a = 41$	Per Third Step
$- e$	15	$y + u = 41 - e$	
$- 13$	16	$u = e - 38$	Hence $e > 38$

From

From the 13th and 16th Steps it appears, that if $a = 22$.
Then $e = 39$. $y = 79 - 2e = 1$. And $u = e - 28 = 1$.
Again.

Suppose	17	$a = 23$	Then	$5a = 115$	And	$3a = 69$
But	18	$2e + y = 189 - 5a = 74$	Per 7th Step.			
18 - 2e	19	$y = 74 - 2e$	Hence	$e < \frac{1}{2} \times 74 = 37$		
Again	20	$e + y + u = 63 - a = 40$	Per 3d Step.			
20 - e	21	$y + u = 40 - e$				
21 - 19	22	$u = e - 34$	Hence	$e > 34$		

From the 19th and 22th Steps it appears, that if $a = 23$,
Then e may be either 35 Or 36.

Once more for a further Illustration.

Let	23	$a = 24$	Then	$5a = 120$	And	$3a = 72$
But	24	$2e + y = 189 - 5a = 69$	Per 7th Step.			
24 - 2e	25	$y = 69 - 2e$	Hence	$e < \frac{69}{2} = 34\frac{1}{2}$		
Again	26	$e + y + u = 63 - a = 39$	Per 3d Step.			
26 - e	27	$y + u = 39 - e$				
27 - 25	28	$u = e - 30$	Hence	$e > 30$		

From hence it appears, that if $a = 24$, Then e may
be either 31 . 32 . 33 . Or 34 . *viz.* it may be any
Number betwixt 30 and $34\frac{1}{2}$ by the 25th, and 28th
Steps, from whence the Values of y and u may easily be
found.

That is, if	{	$e = 31$.	Then	$y = 7$.	And	$u = 1$.
		$e = 32$.		$y = 5$.		$u = 2$.
		$e = 33$.		$y = 3$.		$u = 3$.
		$e = 34$.		$y = 1$.		$u = 4$.

Proceeding on in this Manner with all the other single Values
of a , there may be found above 120 Answers to this Que-
stion in whole Numbers : And if you please to put $a =$ Fra-
ctions, there may be found an Innumerable Set of Answers.
Whereas the Rule of Alligation in Vulgar Arithmetick affords
but only one Answer in Fractions ; to wit, that of $a = 31\frac{1}{2}$.
 $e = 10\frac{1}{2}$. $y = 10\frac{1}{2}$. $u = 10\frac{1}{2}$. As may be easily try'd
per Rule Page 115, &c.

These Two Examples being well understood (especially if
the Last be thoroughly pursu'd) may suffice to shew the Method
of Limiting the Answers to all Sorts of Questions of this kind.
I shall therefore conclude this Chapter of Questions, with gi-
ving a Solution to the Ænigina (or Riddle) Propos'd (by

not Answered) by Mr. John Kersey, in the Close of the Appendix to his *Arithmetick*, which affords several pretty Questions, the Solution whereof will discover a certain Sentence consisting of Three Words, which must be found by the Help of Figures placed (or supposed to be placed) over the Twenty Four Letters of the Alphabet.

Thus $\begin{cases} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \&c. \text{ Called Indices.} \\ a \cdot b \cdot c \cdot d \cdot e \cdot f \cdot g \cdot \&c. \text{ to the Last Letter.} \end{cases}$
So that if the Index of any Letter be once found, the Letter to which it belongs is Consequently known.

The Ænigima.

1. If the Difference between the Indices of the Second Letter of the Second Word; And the Third Letter of the First Word, be Multiplied into the Difference of their Squares, the Product will be 576. And if their Sum be Multiplied into the Sum of their Squares, that Product will be 2336, The Index of the said Third Letter being the Greatest.

Let	1	a = The Greater Index, or that of the 3d Letter.
And	2	e = The Lesser, or that of the Second Letter.
Then {	3	$a - e \times aa - ee = 576$
	4	$a + e \times aa + ee = 2336$
		} By the Question.
3 x	5	$aaa - aae - aee + eee = 576$
4 x	6	$aaa + aae + aee + eee = 2336$
6 - 5	7	$2a ae + 2aee = 1760$
6 + 7	8	$aaa + 3aee + 3aee + eee = 4096$
8 w 3	9	$a + e = \sqrt[3]{4096} = 16$
4 ÷ a + e	10	$aa + ee = \frac{2336}{a + e} = \frac{2336}{16} = 146$
9 @ 2	11	$aa + 2ae + ee = 256$
11 - 10	12	$2ae = 110$
10 - 12	13	$aa - 2ae + ee = 36$
13 w 2	14	$a - e = \sqrt{36} = 6$
9 + 14	15	$2a = 22$
15 ÷ 2	16	$a = 11$
9 - 16	17	$e = 5$

From hence it appears, that the 3d Letter of the 1st Word is l, and the 2d Letter of the 2d Word is e.

Note, In order to set down the Letters (as they become found) in their proper Places, it may be Convenient to supply the vacant Places with Stars.

Thus { First Word Second Word Third Word.
 l ***e*** *****

2. The

2. The *Indices* last found, are the Two *Extreames* of Four Numbers in *Arithmetical Progression*, the *Lesser Mean* being the *Index* of the First Letter of the Third Word; And the *Greater Mean* is the *Index* of the Fourth and Last Letter of the First Word.

Viz. 5 . 7 . 9 . 11 are the Four Terms in *Arith. Progression*. Whence it appears, that G (whose *Index* is 7) is the First Letter of the Third Word; and that i (whose *Index* is 9) is the Fourth or Last Letter of the First Word; which being placed down, will stand thus, ***li. *e***. G*****

3. The Second Letter of the Third Word, is the same with the Third Letter of the First Word; And the Fifth Letter of the Third Word, is the same with the Last Letter of the First Word.

Whence the Letters will stand thus, ***li. *e***. Gl*****

4. The *Sum* of the *Squares* of the *Indices* of the First and Second Letters, of the First Word is 520; And the *Product* of the same *Indices* is seven Ninths of the *Square* of the *Greater Index*, which is the *Index* of the said First Letter.

Let $a =$ the Greater, and $e =$ the Lesser *Index*.

Then	1	$aa + ee = 520$	} According to the Data.
And	2	$ae = \frac{7}{9}aa$	
2 $\div$ a	3	$e = \frac{7}{9}a$	
3 $\odot$ 2	4	$ee = \frac{49}{81}aa$	
1 $-$ 4	5	$aa = 520 - \frac{49}{81}aa$	
5 $\times$ 81	6	$81aa = 42120 - 49aa$	
6 $+$ 49aa	7	$130aa = 42120$	
7 $\div$ 130	8	$aa \frac{42120}{130} = 324$	
8 uv 2	9	$a = \sqrt{324} = 18$	It's Letter is s.
3, 9	10	$e = \frac{7}{9}a = 14$	It's Letter is o.

Hence the Letters will stand thus, *Soli. *e***. Gl*****

5. The *Difference* between the two Last *Indices*, is the *Index* of the First Letter of the 2d Word; *viz.* $18 - 14 = 4$ being the *Index* of the Letter D.

Then the Letters will stand thus, *Soli. De****. Gl*****

6. The

6. The Third and Last Letter of the Second Word, Also the Third Letter of the Third Word, are the same with the Second Letter of the First Word.

Hence the Letters will stand thus, *Soli Deo Glor.*

7. The Sum of the Indices of the Fourth Letter of the Third Word, and the Sixth or Last Letter of the same Word, being Added to their Product, is 35. And the Difference of their Squares is 288. The Index of the Last Letter being the Least. Put a = the Greater, and e = the Lesser Index, As before

Then	1	$ae + a + e = 35$	} By the Data.
And	2	$aa - ee = 288$	
1 — a	3	$ae + e = 35 - a$	
$3 \div a + 1$	4	$e = \frac{35 - a}{a + 1}$	For $e \times a + 1 = ae + e$
4 $\odot$ 2	5	$ee = \frac{1225 - 70a + aa}{aa + 2a + 1}$	
2 + 5	6	$aa = 288 + \frac{1225 - 70a + aa}{aa + 2a + 1}$	
6 $\times$ aa &c.	7	$\{ a^4 + 2a^3 + aa = 288aa + 576a + 288$	
7 $\pm$	8	$+ 1225 - 70a + aa$	
		$a^4 + 2a^3 - 288aa - 506a = 1513$	

This Last Equation being Resolved according to the Method which shall be shewed in the next Chapter, it will be $a = 17$;

its Letter r ; And from the 4th Step $e = \frac{35 - a}{a + 1} = 1$ the Index of the Letter a .

Then these Two Letters being placed according to the Data above, are all that are required by the Enigma to Compleat these Words,

Soli Deo Gloria.

CHAP. X.

The Solution of Affected Equations in Numbers.

Before we proceed to the Solution of Affected Equations, it may not be amiss to shew the Investigation (or Invention) of those Theorems or Rules for Extracting the Roots of Simple Powers, made Use of in Chapter II., Part I.

I shall here make Choice of the same Letters, to represent the Numbers, both given and sought; as in my Compendium of Algebra.

G , always denote the given Resolvend.
 Viz. Let $\begin{cases} r = \text{any Number taken as near the true Root as} \\ \quad \text{may be, whether it be Greater or Less.} \\ e = \text{the unknown Part of the Root sought, by} \\ \quad \text{which } r \text{ is to be either Increased or Decreased.} \end{cases}$

Then if r be any Number Less than the true Root, it will be $r + e =$ the Root sought.

But if r be taken Greater than the true Root, it will then be $r - e =$ the Root sought.

And put D for the Dividend that is produced from G , after it is Lessen'd and Divided by r , &c. (into the Co-efficients of Affected Equations) according as the Nature of the Root requires.

These things being premised, we may proceed to Raising the Theorems.

Section. I.

I. For the Square Root, viz. $aa = G$. Quere a .

Let	1	$r + e = a$	
1 $\ominus$ 2	2	$rr + 2re + ee = aa = G$	
2 $-$ rr	3	$2re + ee = G - rr$	Call it D , viz. $D = G - rr$
Then	4	$\left\{ \frac{D}{2r + e} = e \right\}$	This shews the 1st Method of Extracting the Square Root. Sect. 5. Ch. II. Part I.
3 $\div$ 2	5	$re + \frac{1}{2}ee = \frac{G - rr}{2} = D$	

Which gives this Theorem $\left\{ \frac{D}{r + \frac{1}{2}e} = e \right\}$

The Arithmetical Operations of both these Theorems, you have in the Examples of Section 2. Page 126; To which I refer

refer the *Learner*, supposing him by this *Time* to understand them without any more Words than what is there exprest.

II. To Extract the Cube Root; viz. $aaa = G$. Quere a .

Let	1	$r + e = a$	Supposing r Less than the true Root.
1 $\ominus$ 3	2	$rrr + 3rre + 3ree + eee = aaa = G$	
2 $-$ rrr	3	$3rre + 3ree + eee = G - rrr$	
3 $\div$ 3r	4	$re + ee + \frac{eee}{3r} = \frac{G - rrr}{3r} = D$	

Let $\frac{eee}{3r}$ be Rejected or Cast off, as being of small Value:

Then it will be, $re + ee = D$, which gives this following

Theorem $\left\{ \frac{D}{r + e} = e \right.$

By this Theorem or Rule, the 1st and 2d Examples in Case 1. Page 132. are perform'd; the which being compared with this Theorem, may be very easily understood.

Again, Suppose $aaa = G$, (As before) And let r be taken Greater than the true Root.

Then	1	$r - e = a$	
1 $\ominus$ 3	2	$rrr - 3rre + 3ree = a^3 = G$	$\left\{ \begin{array}{l} eee \text{ being Rejected} \\ \text{as above.} \end{array} \right.$
2 $+$ 3r	3	$3rre - 3ree = rrr - G$	
3 $\div$ 3r	4	$re - ee = \frac{rrr - G}{3r} = D$	

Which gives this Theorem $\left\{ \frac{D}{r - e} = e \right.$

By this Theorem the Third Example in Case 2. Page 133. is perform'd.

III. To Extract the Biquadrate Root; viz. $a^4 = G$. Quere a .

Let	1	$r + e = a$	Supposing r Less than <i>Just</i> .
1 $\ominus$ 4	2	$r^4 + 4rrre + 6rree = a^4 = G$	$\left\{ \begin{array}{l} \text{Rejecting all the} \\ \text{Powers of } e \\ \text{above } ee. \end{array} \right.$
2 $-$ r^4	3	$4rrre + 6rree = G - r^4$	
3 $\div$ 2rr	4	$2re + 3ee = \frac{G - r^4}{2rr} = D$	

Which gives this Theorem $\left\{ \frac{D}{2r + 3e} = e \right.$

By this Theorem the Biquadrat Root of any Number may be Extracted. But as I have already said Page 134. those Extractions may be very well perform'd, by Two Extractions of the Square Root. Vide Example Page 135.

IV. To Extract the Surfsolid Root, viz. $a^3 = G$ Quere a . If r be taken Less than just, then $r + e = a$ As before,

And $\left\{ \frac{G - r^3}{3r^2} = D \right.$ Which gives this Theorem $\left\{ \frac{D}{r + 2e} = e \right.$

By this Theor. the Surfsolid Root Examp. 1. Pag. 136 is Extracted. But if r be taken Greater than just; Then $r - e = a$

And $\left\{ \frac{r^3 - G}{3r^2} = D \right.$ Which gives this Theorem $\left\{ \frac{D}{r - 2e} = e \right.$

By this Last Theorem the Example in Page 137. is perform'd.

I presume it needless to pursue the Raising of these Theorems, for Extracting the Roots of Simple Powers, any further; because the Method of doing it is General, how High soever they are; And therefore it may be easily understood by what is already done.

Section 2.

Notwithstanding I have already shewed the Solution of Quadratick Equations Two several ways, viz. by Casting off the Lowest Term: And by Compleating the Square, vide Sect. 2. Page 195. &c. Yet it may not be amiss to shew, how those Equations may be Resolved into Numbers by this Universal Method of Continued Series; wherein, if the First r be taken Equal to the First true Root or Single Side of the Resolvend; And every Single Value of e (as it becomes found) be still Added to it, for a new r , Then those Roots may be Extracted without repeating a Second Operation, As before in the Single Powers.

Case I. Let $aa + 2ba = G$. 'Tis required to find the Value of a

Put	1	$r + e = a$	
1. $\odot$	2	$rr + 2re + ee = aa$	
1 $\times$ 2b	3	$2br + 2be = 2ba$	
2 + 3	4	$rr + 2br + 2re + 2be + ee = aa + 2ba = G$	
4 - rr &c.	5	$2re + 2be + ee = G - rr - 2br$	
5 $\div$ 2	6	$re + be + \frac{1}{2}ee = \frac{1}{2}G - \frac{1}{2}rr - br = D$	

Which gives this Theorem $\left\{ \frac{D}{r + b + \frac{1}{2}e} = e \right.$

Suppose

Suppose $b = 364$ And $G = 38692865$
 If $r = 6000$ Then $rr = 36000000$ And $2br = 4368000$
 But $36000000 + 4368000 = 40368000 > 38692865 = G$
 Therefore the First $r < 6000$. Let $r = 5000$ Then

1st	$r = 5000$	$19346432,5 = \frac{1}{2}G$	
	$b = 364$	$= 1432000,$	$= \frac{1}{2}rr + br$
1st	$r + b = 5364$	$5026432,5 = D$	$(800 = e$
	$+ \frac{1}{2}e = 400$	46112	
1	Divisor 5764	41523	$(60 = e$
2d	$r + b = 6164$	37164	
	$+ \frac{1}{2}e = 30$	$43592,5$	$(7 = e$
2	Divisor 6194	$43592,5$	$867 = e$
3d	$r + b = 6224$	(0)	
	$+ \frac{1}{2}e = 3,5$		
3	Divisor $6227,5$		

First $r = 5000$ } $= 5867 = a$ As was required.
 $+ e = 867$

Case 2. If $aa - 2ba = G$ Then proceeding as above
 there will arise this Theorem $\left\{ \frac{D}{r - b + \frac{1}{2}e} = e \right.$ &c.

And in Case 3. viz. $2ba - aa = G$ you will have
 this Theorem $\left\{ \frac{D}{b - r - \frac{1}{2}e} = e \right.$ &c. As above.

I think it needless to Trouble the Reader with the Work of
 these Two Theorems in Numbers; because if the last Example of
 Case 1. be understood, the other will be *easier*. Not but that
 the Method of Compleating the Square is very ready and *easier*,
 as you may observe by the Work in several Questions of this
 Chapter.

Section 3.

In the Solution of all Affected Equations, that are above (or
 higher than) Quadratics, it will be the best way to take $r =$
 the next nearest Root of the Equation: And then it will be
 $r + e = a$ if r be Less than *just*; Or $r - e = a$ if r
 be Greater than *just* (as at the beginning of this Chapter.) And
 all the Powers of the unknown Part of the Root (viz. e)
 above its Square (ee) are to be rejected or Cast off; As before
 in

in Raising the Theorems for the Simple Powers. And therefore it is, that to supply the want of those Powers (above ee in the Theorem) the Operation must be repeated: As in the Example of Extracting the Cube Root Page 133. viz. when the Figures in the Root consist of more than Three Places, vide Page 140 and 141.

Suppose $aaa + ba = G$ Quere a ?

Let	1	$r + e = a$ viz. Let r be supposed Less than
1 $\odot$ 3	2	$rrr + 3rre + 3ree = aaa$ (just
1 $\times$ b	3	$br + be = ba$
2 $+$ 3	4	$rrr + br + 3rre + be + 3ree = a^3 + ba = G$
4 $\div$ $3r$	5	$\frac{1}{3}rr + \frac{1}{3}b + re + \frac{be}{3r} + ee = \frac{G}{3r}$
5 $-$ &c.	6	$re + \frac{be}{3r} + ee = \frac{G}{3r} - \frac{1}{3}rr - \frac{1}{3}b = D$

Which gives this Theorem $\left\{ \frac{D}{r + \frac{b}{3r} + e} = e \right.$

But if r be taken Greater than just, Then it will be

$$re + \frac{be}{3r} - ee = \frac{1}{3}rr + \frac{1}{3}b - \frac{G}{3r} = D \quad \text{Which produces}$$

this Theorem $\left\{ \frac{D}{r + \frac{b}{3r} - e} = e \right.$

By either of these Two Theorems the Value of a may be easily found. Or rather otherwise as in the following Example.

Let $aaa + 24a = 587914$ Here $b = 24$
 Suppose the First $r = 90$ Then $r^3 = 729000 > 587914$
 without the 24×90 being Added to it: Therefore $r < 90$
 Again, Suppose $r = 80$ Then $r^3 = 512000$ And $24r = 1920$
 But $512000 + 1920 = 513920 < 587914$ Hence $r > 80$
 but nearer to it than 90. Therefore

it must be

1	$r + e = a$ Less than just
1 $\odot$ 3	2 $rrr + 3rre + 3ree = aaa$
1 $\times$ $\overline{24}$	3 $24r + 24e = 24a$
2, in Num.	4 $512000 + 1920e + 240ee = aaa$
3, in Num.	5 $1920 + 24e = 24a$
4 $+$ 5	6 $513920 + 19224e + 240ee = 587914$
6 $-$ 513920	7 $19224e + 240ee = 73994$
7 $\div$ $\overline{240}$	8 $80,1e + ee = 308,31 = D$
8 $\div$	9 $e = \frac{D}{80,1 + e}$

Operations

Operation $80,1) 308,31 = D \quad (3,7 = e$
 $+ e = 3 \quad 249,3$
 1. Divisor $83,1) 59,01$ First $r = 80,$
 $+ e = ,7 \quad 58,66 \quad + e = 3,7$
 2. Divisor $83,8) ,35 \quad r + e = 83,7$

Or rather New $r = 83,7$ for a Second Operation, which being *involved* and *try'd* (as above) will be found *Greater* than *just* : Therefore

it must be	1	$r - e = a$
1 $\odot 3$	2	$rrr - 3rre + 3ree = aaa$
1 $\times 24$	3	$24e - 24e = 24a$
2, in Num.	4	$586376,253 - 21017,07e + 251,1ee = ada$
3, in Num.	5	$2008,8 - 24e = 24a \quad (914$
4 $+ 5$	6	$588385,053 - 21041,07e + 251,1ee = 587$
6 $\frac{+}{-}$	7	$21041,07e - 251,1ee = 471,053$
7 $\div 251,1$	8	$83,7955e - ee = 1,87595778 = D$
8 $\div$	9	$e = \frac{D}{83,7955 - e}$

2. Operation $83,7955) 1,87595778 \quad (,0223 = e$
 $- e = ,02 \quad 1,675510$
 1. Divisor $83,7755) ,2004477$
 $- e = ,002 \quad ,1675470$
 2. Divisor $83,7735) ,03290078$
 $- e = ,0003 \quad ,02513196$
 3. Divisor $83,7732) ,00776882$

Having once found *half* the Places of *Figures* for the Value of e , it will be needless to form *New Divisors* (as above); for the Rest of the *Figures* may be as truly found by plain *Division* only. Thus

The last Divisor is $83,7732) ,007768820 \quad (,0223 = e$
 $7539588 \quad (,0000927 \}$ Add
 Last $r = 83,7 \quad 2292320 \quad ,0223927 = e$
 $- e = ,0223977 \quad 1675464$
 $r - e = 83,6776073 = a \quad 6168560$
 $5864124 \quad \&c.$

But if more *Exactness* be required, you may make the *New* $r = 83,6776073$ And proceed with it to a *Third Operation*;

ration; which will afford Twenty Seven Places of Figures for the Value of a . That is, every Operation will produce Triple the Places of Figures to those of the precedent r . And this Tripling the Places of Figures in the Root, at every Operation, holds Good and is to be Observed in the Solution of all Adfected Equations (how high soever they are) according to this Method of Resolving them. See Page 141.

Example 2. Suppose $aaa - ba = G$ Quere a .

If $r + e = d$ Then $re - \frac{\frac{1}{3}be}{r} + ce = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$

which gives this Theorem $\left\{ \frac{D}{r - \frac{\frac{1}{3}b}{r} + e} = e \right.$

But if $r - e = a$ Then $re + \frac{\frac{1}{3}be}{r} + ce = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$

which gives this Theorem $\left\{ \frac{D}{r + \frac{\frac{1}{3}b}{r} + e} = e \right.$

Or you may proceed otherwise, as in the Last Example.

Let $aaa - 6438a = 104785688$ Here $b = 6438$

Suppose the First $r = 500$. $rr = 125000000$ and $br = 3219000$

Then $125000000 - 3219000 = 121781000$

But $121781000 > 104785688$ Therefore $r < 500$

Again, Suppose $r = 400$. $rrr = 64000000$ and $br = 2575200$

Then will $64000000 - 2575200 = 6142800$

But $6142800 < 104785688$ Hence $r > 400$

Consequently r is betwixt 400 and 500, But 500 is the next nearest; Therefore, Let $r = 500$ being Greater than just.

Then	1	$r - e = a$
1 $\odot$ 3	2	$rrr - 3rre + 3ree = aaa$
1 $\times$ 6	3	$br - be = ba$
2, in Num.	4	$125000000 - 7500000 + 150000 = aaa$
3, in Num.	5	$3219000 - 6438e = 6438a \quad (5688$
4 $-$ 5	6	$121781000 - 743562e + 150000 = 10478$
6 $+$	7	$743562e - 150000 = 16995312$
7 $\div$ 1500	8	$495e - ce = 11330 = D$
8 $\div$	9	$e = \frac{D}{495 - c}$

Operation

Operation 495) 11330 (23,8 = e
 $-e = 20$ 950

1. Divisor 475) 1830
 $-e = 3$ 1416

2. Divisor 472) 414,0
 377,6

First $r = 500$
 $-e = 23,8$

$r - e = 476,2 = a$

Let New $r = 476$ for a 2d Operation. Then $r^3 = 107850176$
 and $br = 3064488$ But $107850176 - 3064488 = 104785688$
 the same with the Resolvend. Consequently $a = 476$ just.

Example 3. Let $ba - aaa = G$. Quere a .

If $r + e = a$ Then $\frac{\frac{1}{3}be}{r} - re - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$

which gives this Theorem $\left\{ \frac{D}{\frac{\frac{1}{3}b}{r} - r - e} = e \right.$

But if $r - e = a$ Then $re - \frac{\frac{1}{3}be}{r} - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$

which gives this Theorem $\left\{ \frac{D}{r - \frac{\frac{1}{3}b}{r} - e} = e \right.$

Or otherwise as before in the Two Last Examples. Thus

Let $123456a - aaa = 12272861$. Here $b = 123456$.
 Suppose the First $r = 200$ Then $rrr = 8000000$. and
 $br = 24691200$. then $24991200 - 8000000 = 16691200$
 but $16691200 > 12272861$. therefore r is here Less than
 just, because the highest Power is — or Negative.

Again, Suppose $r = 300$ then $r^3 = 27000000$ and $br = 37036800$
 Then $37036800 - 27000000 = 10036800 < 12272861$
 Consequently $r < 300$ and $r > 200$.

Let $r = 300$. being the next nearest, but more than just.

Then	1	$r - e = a$
1 $\odot$ 3	2	$rrr - 3rre + 3ree = aaa$
1 $\times$ b	3	$br - be = ba$
2, in Numb.	4	$27000000 - 2700000e + 900ee$
3, in Numb.	5	$37036800 - 123456e$
5 — 4	6	$10036800 + 146544e - 900ee = 12272861$
6 —	7	$146544e - 900ee = 2236061$
7 $\div$ 900	8	$162e - ee = 2484 = D$
1 $\div$ &c.	9	$e = \frac{D}{162 - e}$

I i Operatio

Operation. $162) \quad 2484 \quad (16,6 = e$

$$- e = 10 \quad 152$$

1. Divisor $152) \quad 964$

$$- e = 6 \quad 876$$

2. Divisor $146) \quad 88,0$

$$87,6$$

First $r = 300$

$$- e = 16,6$$

$$r - e = 283,4 = a$$

Or New $r = 283$ which being Involved, &c. will appear to be the true Root. That is, $a = 283$ just.

Note, These are usually Called the Three Forms of Cubick Equations; and in the Solution of the Third or Last Form, viz. $ba - aaa = G$, you may meet with some seeming Difficulties; especially in making Choice of the First r because this Equation is An Ambiguous Equation, and hath Two Affirmative Roots, viz. a Greater and Lesser Root. But having once found either of them, the other may be easily obtained by Division only; as in the Quadratick Equations. Vide Chapter 8.

As for Instance, in the Last Example $a = 283$

And $123456a - aaa = 12272861$. Make these Two Equations $= 0$. To wit, Let $a - 283 = 0$.

And $-aaa + 123456a - 12272861 = 0$

Then, $a - 283) -aaa + 123456a - 12272861$ ($-aa$

$$-aaa + 283aa$$

$$- 283aa + 123456a$$

$$(- 283a$$

$$- 283aa + 80089a$$

$$+ 43367a - 12272861$$

$$(+ 43367$$

$$+ 43367a - 12272861$$

$$(0)$$

$$(0)$$

Hence it appears, that $-aa - 283a + 43367 = 0$ Consequently $aa + 283a = 43367$ this Equation being Solved, $a = 110,2722$ &c. which is the Lesser Root of the aforesaid Equation $ba - aaa = G$ &c.

After this manner all the possible, and impossible Roots of any Equation may be easily discovered, any one of its Roots being once found. I shall therefore omit inserting more Examples of that kind.

Suppose $aaa + baa + ca = G$: Quere a .

Let $b = 74$, $c = 8729$, and $G = 560783$

By Trials (as before) it will be found that the next nearest $r = 40$ being something Less than just.

Therefore

Therefore	1	$r - e = a$
1 $\times c$	2	$cr + ce = ca$
1 $\odot 2 : \times b$	3	$brr - 2bre - bee = baa$
1 $\odot 3$	4	$rrr - 3rre - 3ree = aaa$
2, in Numb.	5	$349160 + 8729e$
3, in Numb.	6	$118400 + 5920e + 74ee$
4, in Numb.	7	$64000 + 4800e + 120ee$
5 + 6 + 7	8	$531560 + 19449e - 194ee = 560783$
8 - 531560	9	$19449e - 194ee = 29223$
9 $\div 194$	10	$100,2e + ee = 153,06 = D$
10 $\div$	11	$e = \frac{D}{100,2 + e}$

Operation.	100,2	153,06	(1,5 = e
+ e =	1	101,2	First r = 40
1. Divisor	101,2)	51,86	+ e = 1,5
+ e =	,5	50,85	r + e = 41,5 = r
2. Divisor	101,7	1,01	

Or New $r = 41,5$ for a Second Operation, which being duely Involved, &c. will be found more than just.

Therefore	1	$r - e = a$
Then {	2	$cr - ce = ca$
	3	$brr - 2bre + bee = baa$
	4	$rrr - 3rre - 3ree = aaa$

These being turn'd into Numbers, &c. As above, they will be $20037,75e - 198,5ee = 390,375$ which being Divided by 198,5 the Co-efficient of ee will become $100,946e - ee = 1,966624$ &c. = D

Operation	100,946	1,966624	(,019 = e
- e =	,01	100936	
1. Divisor	100,936)	957264	
- e =	,009	908343	
2. Divisor	100,927)	* 489210	(,0004847
* Here I proceed by		403708	,0194847 = e
plain Division, without		855020	
forming New Divisors.		807416	
Last r = 41,5		476040	
- e =	,0194847	403708	
r - e =	41,4805153 = a	723320	&c.

Let the Last *Æquation* in the *Ænigma*, Chap. 9. be here proposed for a Solution.

$$\text{Viz. } aaaa + baaa - caa - da = G$$

$$b = 2 : c = 288 : d = 506 : \text{ and } G = 1513 \text{ Quiere } a$$

By Tryals it will be found, that the next nearest $r = 20$ being something More than just.

Therefore	1	$r - e = a$
1 $\times$ d	2	$dr - de = da$
1 $\odot$ 2: $\times$ c	3	$crr - 2cre + cee = caa$
1 $\odot$ 3: $\times$ b	4	$brrr - 3brre + 3bree = baaa$
1 $\odot$ 4	5	$r^4 - 4rrre + 6rree = aaaa$

These being turned into Numbers, and those duely Collected, according as the Signs of the *Æquation* direct, they will become

$$50680 - 22374e + 2232ee = 1513. \text{ which being all Divided by } 2232 \text{ the Co-efficient of } ee, \text{ will become } 10e - ee = 22 = D.$$

$$\text{Then } \left\{ \frac{D}{10 - e} = e \right.$$

Operation,	10)	22	(3 = e
- e =	3	21	
Divisor	7)	1	

$$\text{First } r = 20$$

$$- e = 3$$

$$r - e = 17 = a \text{ just.}$$

See the End of Chap. 9.

By what hath been already done, about the Solution of these few *Æquation* (being carefully Observed,) I presume the Learner will easily Conceive how to proceed in the Solution of all kind of *Æquations*, be they never so High or Adfected; therefore I shall not here propose many various Examples, but only take them as they fall in Course when I come to the next Part, wherein you will (perhaps) find such *Æquations* with their Solutions as are not common.

C H A P. XI.

Of Simple Interest, and Annuities or Pensions, &c.

Interest or the Use paid for the Loan of Money may be either Simple; Or Compound.

Section 1. Of Simple Interest.

Simple Interest, is that which is paid for the Loan of any Principal or Sum of Money, Lent out for some Time, at any Rate per Cent. agreed on between the Borrower and the Lender; which, according to the Laws of England, ought to be but Six Pounds for Use of 100l. for one Year, and Twelve Pounds for Use of 100l. for Two Years. And so on for a Greater or Lesser Sum, proportionable to the Time proposed.

There are several Ways of Computing (or Answering Questions about) Simple Interest; As by the Single and Double Rule of Three (See Page 96. &c.) Others make Use of Tables Composed at several Rates per Cent. As Sir Samuel Morland in his Doctrine of Interest both Simple, and Compound, is all perform'd by Tables; wherein he hath detected several material Errors Committed by Doctor Newton, Mr. Kersey upon Wingate, and Mr. Clavil, &c. in the Business of Computing Interest, &c. by their Tables, too tedious to be here repeated.

But I shall in this Tract take other Methods, and shew that all Computations relating to Simple Interest, are grounded upon Arithmetick Progression; and from thence Raise such General Theorems, as will suit with all Cases. In order to that

Let $\begin{cases} P = \text{Any Principal or Sum put to Interest.} \\ R = \text{The Ratio of the Rate, per Cent. per Annum.} \\ t = \text{The Time of the Principal's Continuance at Interest.} \\ A = \text{The Amount of the Principal, and its Interest.} \end{cases}$

Note, the Ratio of the Rate, is only the Simple Interest of 1l. for one Year, at any given Rate; and its thus found,

Viz. $100 : 6 :: 1 : 0,06 = \text{the Ratio at 6 per Cent. per An.}$

Or $100 : 7 :: 1 : 0,07 = \text{the Ratio at Seven per Cent. \&c.}$

Again $100 : 7,5 :: 1 : 0,075 = \text{the Ratio at 7 and } \frac{1}{2} \text{ per C.}$

And if the given Time be whole Years; then $t =$ the Number of those Years: But if the Time given, be either pure Parts of a Year, Or Parts of a Year Mix'd with Years, those Parts must be turn'd into Decimals; and then $t =$ those Decimals, &c. Now the Common Parts of a Year may be easily

easily turn'd or converted into *Decimal Parts*, if it be considered

That One $\left\{ \begin{array}{l} \text{Day is the } \frac{1}{365} \text{ Part of a Year} = 0,00274 \text{ fere.} \\ \text{Month is the } \frac{1}{12} \text{ Part of a Year} = 0,083333 \text{ \&c.} \\ \text{Quarter is the } \frac{1}{4} \text{ Part of a Year} = 0,25 \\ \text{Half a Year} = 0,5 \quad \text{And Three Quarters} = 0,75 \end{array} \right.$

These things being premised, we may proceed to *Raising the Theorems*.

Let R = the Interest of 1*l.* for one Year. As before.

Then $2R$ = the Interest of 1*l.* for two Years.

And $3R$ = the Interest of 1*l.* for three Years.

$4R$ = the Interest of 1*l.* for four Years. And so on for any Number of Years proposed.

Hence it is plain, that the *Simple Interest* of one Pound, is a *Series of Terms* in *Aritbmectick Progression* Increasing; whose first Term and Common Difference is R . And the Number of all the Terms is t . Therefore the Last Term will always be tR = the Interest of 1*l.* for any given Time signified by t .

Then $\left\{ \begin{array}{l} \text{As one Pound : Is to the Interest of 1*l.* :: So is any} \\ \text{Principal or given Sum : To its Interest.} \end{array} \right.$

That is, $1 : tR :: P : tRP$ = the Interest of P . Then the Principal being Added to its Interest, their Sum will be = A the Amount required : Which gives this General Theorem.

$$\text{Theorem } tRP + P = A.$$

From whence the Three following Theorems are easily deduced.

$$\text{Theorem 2. } \left\{ \frac{A}{tR + 1} = P. \quad \text{Theorem 3. } \left\{ \frac{A - P}{tP} = R \right. \right.$$

$$\text{Theorem 4. } \left\{ \frac{A - P}{RP} = t \right.$$

These Four Theorems Resolve all Questions about *Simple Interest*.

Question 1. What will 256*l.* 10*s.* Amount to in 3 Years, one Quarter, 2 Months and 18 Days, at 6 per Cent. per Annum,

Here is given $P = 256,5$. $R = 0,06$. And $t = 3,46599$

For 3 Years = 3, Quere A . Per Theorem 1.

one Quater = 0,25

2 Months = 0,16667 = 0,08333 $\times$ 2

18 Days = 0,04932 = 0,00274 $\times$ 18

Hence $t = 3,46599 : \times 0,06 = 0,2079594 = tR$

Then $0,2079594 \times 256,5 = 53,341586 = tRP$

And $53,341586 + 256,5 = 309,841586 = tRP + P = A$

That is, 309*l.* 16*s.* 10*d.* being the Answer required.

Question

Question 2. *What Principal or Sum being put to Interest, will Raise a Stock of 309l. 16s. 10d. in 3 Years, one Quarter, Two Months and 18 Days; at 6 per Cent. per Annum?*

Or the same Question otherwise stated thus.

What is 309l. 16s. 10d. due 3 Years, one Quarter, 2 Months and 18 Days hence, worth in ready Money; Abating or Discounting 6 per Cent. &c.

Here is given $A = 309,841586$ $R = 0,06$ $t = 3,46599$
(found as before) Thence to find P . Per Theorem 2.

First $3,46599 \times 0,06 = 0,2079594 = tR$

Then $tR + 1 = 1,2079594$ $309,841586 = A (256,5 = P$
That is, $256,5 = 256l. 10s.$ the Answer required.

Question 3. *At what Rate of Interest, per Cent. &c. will 256l 10s. Amount to 309l. 16s. 10d. In 3 Years, one Quarter, 2 Months and 18 Days.*

Here is given, $P = 256,5$ $A = 309,841586$ and $t = 3,46599$
To find R . Per Theorem 3.

First $309,841586 - 256,5 = 53,341586 = A - P$

Next $3,46599 \times 256,5 = 889,026435 = tR$

And $tR = 889,026435$ $53,341586$ ($0,06 =$ the Ratio.

Then $11 : 0,06 :: 100 : 6 =$ the Rate required.

Question 4. *In what Time will 256l. 10s. Raise a Stock of (or Amount to) 309l. 16s. 10d. at 6 per Cent. &c.*

Here is given, $P = 256,5$ $A = 309,841586$ and $R = 0,06$
To find t . Per Theorem 4.

First $309,841586 - 256,5 = 53,341586 = A - P$

And $256,5 \times 0,06 = 15,39 = PR$.

Then $15,39$ $53,341586$ ($3,46599 = t$

That is $t = 3$ Years and $,46599$ Decimal Parts of a Year;
which may be brought into Common Parts of a Year, thus

$0,46599$ And $0,08333$ $0,21599$ (2 Months.
 $- 0,25 =$ one Quarter $,16666$

$0,21599$ $0,02074$ $,04933$ (18 Days.

Hence $t = 3$ Years, one Quarter, 2 Months and 18 Days;
the Answer required.

It must needs be easie to Conceive, that what is here done at 6 per Cent. may be done at any other Rate of Interest, by forming the Ratio, viz. R , accordingly.

Scholium

Scholium.

Altho' it be according to the Laws and Custom of *England*, to Compute Interest at the Proportion of 6 per Cent. (as above) yet he that takes up Money at Interest for any Time Less than Even or Compleat Years, pays more Interest than seems reasonably Due, according to the true Rules of Art.

As for Instance ; If 100*l.* be forborn at Interest one Whole Year, it Amounts to 106. But (I say) if it be paid at the Half Year's End, it should not Amount to 103*l.* As appears from this following Proportion.

Let a = the Amount due at the Half Year's End ; Then it will be $100 : a :: a : 106$ the Amount at the Year's End. Ergo $aa = 10600$ And $a = \sqrt{10600} = 102,9563 = 102*l.* 19*s.* 1½*d.*$ which is Less than 103*l.* by 10½*d.* And if it be paid in Less than Half a Year's Time, the Error must needs be the Greater.

Section 2. Of Annuities, or Pensions in Arrears ; Computed at Simple Interest.

Annuities or Pensions, &c. are said to be in *Arrears*, when they are payable or Due, either *Yearly* or *Half-yearly*, &c. and are Unpaid for any Number of Payments. Therefore the Business is, to compute what all those Payments will Amount unto, allowing any Rate of Simple Interest for their Forbearance, from the Time each particular Payment became Due : Now in order to that,

Put $\begin{cases} u = \text{the Annuity, Pension, or Yearly Rent, \&c} \\ t = \text{the Time of its Continuance (or being) Unpaid.} \\ R = \text{the Ratio, or Interest of 1*l.* for 1 Year, As before.} \\ A = \text{the Amount of the Annuity and its Interest.} \end{cases}$

Then if u = the First Years Rent, due without Interest.

And $\begin{matrix} Ru = \text{the Interest} \\ 2u = \text{the Rent} \end{matrix} \left. \vphantom{\begin{matrix} Ru \\ 2u \end{matrix}} \right\} \text{Due at the End of the Second Year.}$

And $\begin{matrix} 2Ru = \text{the Interest} \\ 3u = \text{the Rent} \end{matrix} \left. \vphantom{\begin{matrix} 2Ru \\ 3u \end{matrix}} \right\} \text{Due at the End of the Third Year.}$

And $\begin{matrix} 3Ru = \text{the Interest} \\ 4u = \text{the Rent} \end{matrix} \left. \vphantom{\begin{matrix} 3Ru \\ 4u \end{matrix}} \right\} \text{Due at the End of the Fourth Year.}$

And $\begin{matrix} 4Ru = \text{the Interest} \\ 5u = \text{the Rent} \end{matrix} \left. \vphantom{\begin{matrix} 4Ru \\ 5u \end{matrix}} \right\} \text{Due at the End of the Fifth Year.}$

And so on for any Number of Years. Hence it's Evident, that $Ru + 2Ru + 3Ru + 4Ru + 5u = A$ the Sum of all the Rents and their Interest, being forborn 5 Years. From whence

it follows, that $Ru + 2Ru + 3Ru + 4Ru = A - tu$.

Here $t=5$. Divide by u , Then $R+2R+3R+4R = \frac{A-tu}{u}$

Next to find the Sum of this Progression (See Page 185.) Thus

Let $R+2R+3R+4R$ &c. $=z$. Then $1+2+3+4$ &c. $=\frac{z}{R}$

Here the Sum of the First and Last Terms are $4+1=5=t$
And the Number of all the Terms is $4=t-1$. Therefore

$\frac{t-1}{2} \times t =$ the Sum of all the Terms. That is, $\frac{tt-t}{2} = \frac{z}{R}$

Hence $\frac{ttR-tR}{2} = z$. Consequently $\frac{ttR-tR}{2} = \frac{A-tu}{u}$.

Now from this Equation it will be easie to deduce the following Theorems.

Theorem 1. $\left\{ \frac{ttRu-tRu+2tu}{2} = A \right.$ Or $\left\{ \frac{ttu-tu}{2} \times R : +tu = A \right.$

Theorem 2. $\left\{ \frac{2A}{ttR-tR+2t} = u \right.$ Theorem 3. $\left\{ \frac{2A-2tu}{ttu-tu} = R \right.$

Let $\frac{2}{R} - 1 = x$. Then $t = \sqrt{\frac{2A}{Ru} + \frac{xx}{4}} : -\frac{1}{2}x \}$ Theor. 4.

Question 1. If 250l. Yearly Rent (or Pension, &c.) be forborn or unpaid Seven Years; What will it Amount to in that Time, at 6 per Cent. for each Payment as it becomes Due?

Here is given $u=250$. $t=7$. And $R=0,06$ To find A . Per
Th. 1. First $250 \times 7 = 1750 = tu$. $1750 \times 7 = 12250 = ttu$
Again $12250 - 1750 = 10500 = ttu - tu$. And $\frac{10500}{2} \times 0,06 = 315$
Lastly $315 + 1750 = 2065 = A$. Viz. 2065l. is the Ans. requir'd.

But if the Annuity, Rent or Pension, is to be paid by Quarterly or Half Yearly Payments, &c.

Then $\frac{0,06}{2} = 0,03 = R$ for Half Yearly Payments.

And $\frac{0,06}{4} = 0,015 = R$ for Quarterly; Or $0,045 = R$ for Three Quarterly Payments. Example of Half Yearly Payments.

Suppose 250l. per Annum, to be paid by Half Yearly Payments, are in Arrears or unpaid for Seven Years; What would it Amount to, allowing 6 per Cent. per Annum for each Payment as it becomes Due.

In this Example there is given $u=125 = \frac{250}{2}$. $t=14$ the Number of Payments; And $R=0,03 = \frac{0,06}{2}$. Thence to find A .

First $125 \times 14 = 1750 = tu$. $1750 \times 14 = 24500 = ttu$.
Again $24500 - 1750 = 22750 = ttu - tu$. Then $\frac{22750}{2} = 11375$

K k

And

And $11375 \times 0,03 = 341,25$ Lastly $341,25 + 1750 = 2091,25$
That is, $A = 2091\text{ l. } 5\text{ s.}$ the Answer required.

N. B. Hence it may be Observed, that Half Yearly Payments,
are more Advantagious than Yearly :

For $2091\text{ l. } 5\text{ s.} > 2065\text{ l.}$ by $26\text{ l. } 5\text{ s.}$ Consequently,
Quarterly Payments are more Advantagious than Half
Yearly Payments.

Question 2. What Yearly Rent, Pension, &c. being forborn
or unpaid Seven Years, will Raise a Stock of 2065 l. allowing 6 per
Cent. per Annum for each Payment as it becomes Due?

Here is given $A = 2065$. $t = 7$ And $R = 0,06$. To find u .
Per Theorem 2.

First $7 \times 0,06 = 0,42 = tR$. and $0,42 \times 7 = 2,94 = ttR$.

Then $ttR - tR = 2,52$

Lastly $ttR - tR + 2t = 16,52$) $4130 = 2A$ ($250 = u$.
That is, 250 l. per Annum , &c. will Raise 2065 l. the Stock
required.

Question 3. In what Time will 250 l. Yearly Rent, Raise a
Stock of 2065 l. Allowing 6 per Cent. &c. for the Forbearance
of the Payments, as they become Due?

Here is given $u = 250$. $A = 2065$ And $R = 0,06$ To find t .
Per Theorem 4. First

$\frac{2}{R} = \frac{2}{0,06} = 33,3333$ And $33,3333 - 1 = 32,3333 = x = \frac{2}{R} - 1$

Then $16,1666 \text{ \&c.} = \frac{1}{2}x$. $261,3605 \text{ \&c.} = \frac{1}{4}xx$.

Again $\frac{4130}{13} = 275,333 = 2A \div Ru$ And $275,3333 + 261,3605$

$= 536,6938 = \frac{2A}{Ru} + \frac{1}{4}xx$ Then $\sqrt{536,6938} = 23,1666$

Lastly $23,1666 - 16,1666 = 7 = t$ the Time required:

Question 4. If 250 l. Yearly Rent, being forborn Seven Years
will Amount to 2065 l. allowing Simple Interest for every Payment
as it becomes Due; What must the Rate of Interest be per Cent. &c.

Here is given $u = 250$. $A = 2065$. And $t = 7$. To find R .
Per Theorem 3.

Thus $\begin{cases} ttu = 12250 . & 4130 = 2A \\ -tu = 1750 . & 3500 = 2tu \end{cases}$

$ttu - tu = 10500$) $630 = 2A - 2tu$ ($0,06 = R$

Then $1 : 0,06 :: 100 : 6$ the Rate required.

Section 3. The Present Worth of Annuities or Pensions, &c. Computed at Simple Interest.

The Business of Purchasing Annuities, or taking of Leases, &c. for any assigned Time, Depends upon the true Equating of the Principal or Money laid out on the Purchase, with the Annuity or Yearly Rent, by allowing (or Discounting) the same Rate of Interest to both Parties. Which may be easily perform'd by duely Applying the respective Theorems of the Two Last Sections together. As will fully appear by the following Question.

Question 1. What is 75^l. Yearly Rent, to continue Nine Years, worth in ready Money, at 6 per Cent. per Annum Simple Interest?

1. Per Theorem 1. of the Last Section, find what the proposed Yearly Rent would Amount to, if it were forborn 9 Years at 6 per Cent.

Thus $u = 75$. $t = 9$. And $R = 0,06$. Quere A

$$ttu = 6075$$

$$tu = 675$$

$$ttu - tu = 5400$$

$$\text{Then } 2) \begin{array}{r} 5400 \quad (2700 \\ R = 0,06 \end{array} \left. \vphantom{\begin{array}{r} 5400 \\ R = 0,06 \end{array}} \right\} \text{Multiply}$$

$$\begin{array}{r} 162, \\ + tu = 675 \end{array} \left. \vphantom{\begin{array}{r} 162, \\ + tu = 675 \end{array}} \right\} = 837 = A$$

2. Then by Theorem 2. Section 1. find what Principal, being put to Interest for the same Time, and at the same Rate, will Amount to 837^l. = A

Thus $tR = 0,54 = 9 \times 0,06$. $tR + 1 = 1,54$ $837 (543,5064 = P$
That is, $P = 543^l. 10^s. 1^d. Which is the Worth of 75^l. a Year, as was required.$

From the Work of these Two Operations, (duely Consider'd) it must needs be Easie to Conceive, how the Two Theorems by which they were perform'd, may be Combined into One.

$$\text{For 1. } \frac{ttRu - tRu + 2tu}{2} = A. \text{ And 2. } PtR + P = A,$$

Consequently $PtR + P = \frac{ttRu - tRu + 2tu}{2}$. And from this Equation, may be deduced the following Theorems.

$$\text{Theorem 1. } \left\{ \frac{ttRu - tRu + 2tu}{2tR + 2} = P. \text{ Or } \left\{ \frac{ttR - tR + 2t}{2tR + 2} : xu = P \right. \right.$$

By this Theorem all Questions of the same kind with the Last viz. that above) may be easily and readily Answered at one Operation.

Theorem 2. $\left\{ \frac{2PtR + 2P}{ttR - tR + 2t} = u. \text{ Or } \left\{ \frac{tR + 1}{ttR - tR + 2t} : \times 2P = u \right. \right.$

Theorem 3. $\left\{ \frac{2P - 2tu}{ttu - tu - 2Pt} = R. \right.$

Let $\frac{2}{R} - \frac{2P}{u} - 1 = x.$ Then will $tt \pm xt = \frac{2P}{Ru}$

Which gives this Theorem 4. $\left\{ \sqrt{\frac{2P}{Ru} + \frac{xx}{4}} : \pm \frac{x}{2} = t. \right.$

By the Second and Third Theorems, Two very Useful Questions may be easily Answered.

1. As for Instance: If it be required to find what Annuity, or yearly Rent &c. may be purchased, for any proposed Sum, to continue any assigned Time, allowing any Rate of Interest.

This Question may be Answered by Theorem 2.

2. Again: If it be required to find how Long any Yearly Rent, Pension, or Annuity &c. may be purchased (or enjoy'd) for any proposed Sum, at any given Rate of Interest.

All Questions of this kind are easily Answered per Theorem 4.

In these Questions it is supposed, that the Purchase or Yearly Rent, is to Commence or be immediately Enter'd upon. But if it be required to find the Value or Purchase of any Annuity or Yearly Rent, &c. in Reversion; That is, when it is not to be Enter'd upon until after some Time, or Number of Years are past; Then you must first find what the Sum propos'd to be laid out in the Purchase, would Amount to, if it were put to Interest, during the Time the Annuity; &c. is not to be in present Possession; And make that Amount the Sum for the Purchase, proceeding with it as in either of the Two Last Questions, &c.

Note, From the First Question of this Section it will be easy to Conceive how to perform the Equating of Payments, between Debtor and Creditor, at any Rate of Interest, without doing any Damage to either Party.

That is, when several Sums of Money are to be paid, at several different Times, To find the Time when all the Payments may be truly discharged at once: As if one Sum were to be paid at the end of 2 Months, another at 6 Months, and perhaps a Third Sum at 8 Months End, &c. And it were required to find the Time when all those Sums may be truly discharged at one Payment without Loss, &c.

C H A P

C H A P. XII.

Of Compound Interest, and Annuities, &c.

Compound Interest is that which arises from any *Principal* and its *Interest* put together, as the *Interest* still becomes Due; So that at every *Payment*, or at the *Time* when the *Payments* became Due, there is Created a *New Principal*; And for that Reason its Called *Interest* upon *Interest*, or *Compound Interest*.

As for Instance; Suppose 100*l.* were Lent out for Two Years, at 6 per Cent. per Annum, Compound Interest: Then, at the End of the First Year, it will only Amount to 106*l.* As in Simple Interest. But for the Second Year this 106*l.* becomes Principal, which will Amount to 112*l.* 7*s.* 2½*d.* at the Second Year's End, whereas by Simple Interest it would have Amounted to but 112*l.*

And altho' it be not Lawful to Lett out Money at Compound Interest; Yet in Purchasing of Annuities or Pensions, &c. And taking Leases in Reversion, it is very usual to allow Compound Interest to the Purchaser for his ready Money; and therefore it is very requisite to understand it.

Section 1. Of Compound Interest.

Let $\left\{ \begin{array}{l} P = \text{the Principal put to Interest.} \\ t = \text{the Time of its Continuance.} \\ A = \text{the Amount of the Principal and Interest.} \\ R = \left\{ \begin{array}{l} \text{the Amount of 1*l.* and its Interest for 1 Year at} \\ \text{any given Rate, which may be thus found.} \end{array} \right. \end{array} \right. \right\} \text{As before}$

Viz. 100 : 106 :: 1 : 1,06 = the Amount of 1*l.* at 6 per Cent.
Or 100 : 107 :: 1 : 1,07 = the Amount of 1*l.* at 7 per Cent.
and so on for any other assigned Rate of Interest.

Then if R = the Amount of 1*l.* for One Year, at any Rate.

RR = the Amount of 1*l.* for Two Years.

RRR = the Amount of 1*l.* for Three Years.

R^4 = the Amount of 1*l.* for Four Years.

R^5 = the Amount of 1*l.* for Five Years. Here $t = 5$

For $1 : R :: R : RR :: RR : RRR :: RRR : R^4 :: R^4 : R^5$ &c. in $\therefore$

That is $\left\{ \begin{array}{l} \text{As one Pound : Is to the Amount of one Pound at} \\ \text{one Year's End :: So is that Amount : To the} \\ \text{Amount of one Pound at Two Year's End, \&c.} \end{array} \right.$

Whence

Whence it is plain, that Compound Interest is grounded upon a Series of Terms, Increasing, in Geometrical Proportion Continued; wherein t (*viz.* the Number of Years) does always assign the *Index* of the last and highest Term, *viz.* the Power of R , which is R^t .

Again, As $1 : R^t :: P : PR^t = A$ the Amount of P for the Time, that $R^t =$ the Amount of 1 .

That is $\left\{ \begin{array}{l} \text{As one Pound : Is to the Amount of one Pound for} \\ \text{any given Time :: So is any proposed Principal (or} \\ \text{Sum : To its Amount for the same Time.} \end{array} \right.$

From the Premises, (*I presume*) the Reason of the following Theorems, may be very Easily understood.

Theorem 1. $PR^t = A$ As above.

From hence the Two following Theorems are easily deduced.

Theorem 2. $\left\{ \frac{A}{R^t} = P \right.$ Theorem 3. $\left\{ \frac{A}{P} = R^t \right.$

By these Three Theorems, all Questions about Compound Interest, may be truly Resolved by the Pen only, *viz.* without Tables; Tho' not so readily as by the help of Tables, Calculated on Purpose; As will appear farther on.

Question 1. What will 256*l.* 10*s.* Amount to, in Seven Years, at 6 per Cent. per Annum, Compound Interest.

Here is given $P = 256,5$. $t = 7$. and $R = 1,06$ which being Involved until its Index $= t$ (*viz.* 7) will become $R^7 = 1,50363$

Then $1,50363 \times 256,5 = 385,6811 = A = 385*l.* 13*s.* 7½*d.*$ which is the Answer required.

Question 2. What Principal or Sum of Money, must be put (or Lett) out, to Raise a Stock of 385*l.* 13*s.* 7½*d.* in Seven Years, at 6 per Cent. per Annum Compound Interest?

Here is given $A = 385,6811$ $R = 1,06$ and $t = 7$ To find P , by Theorem 2.

Thus $R^t = 1,50363$) $385,6811 = A$ ($256,5 = P$.

That is, $P = 256*l.* 10*s.*$ which is the Principal or as Sum was required.

Question

Question 3. In what Time will 256l. 10s. Raise a Stock of (or Amount to) 385l. 13s. 7½d. allowing 6 per Cent. per Annum Compound Interest?

Here is given $P=256,5$ $A=385,6811$ $R=1,06$ To find t by the Third Theorem $\left\{ R^t = \frac{A}{P} = \frac{385,6811}{256,5} = 1,50363 \right\}$, which being continually Divided by $R=1,06$ until nothing Remain, the Number of those Divisions will be $= 7 = t$.

Thus $1,06 \times 1,50363 (1,41852$ And $1,06 \times 1,41852 (1,338225$ Again $1,06 \times 1,338225 (1,262477$. And so on until it become $1,06 \times 1,06 (1$. which will be at the Seventh Division. Therefore it will be $t=7$ the Number of Years required by the Question.

Question 4. If 256l. 10s. will Amount to (or Raise a Stock of) 385l. 13s. 7½d. in Seven Years Time; What must the Rate of Interest be, per Cent. per Annum.

Here is given $P=256,5$ $A=385,6811$ and $t=7$ Quere R .

By Theorem 3. $\left\{ \frac{A}{P} = R^t = 1,50363 \right\}$. As before in the Last Question. And if $R^t = R^7 = 1,50363$ Then $R = \sqrt[7]{1,50363}$ which may be thus Extracted.

Put	1	$r + e + R$	Then
1 $\odot$ 7	2	$r^7 + 7r^6e + 21r^5ee = R^7 = 1,50363 = G$	
2 $- r^7$	3	$7r^6e + 21r^5ee = G - r^7$	
3 $\div 7r^5$	4	$re + 3ee = \frac{G - r^7}{7r^5} = D$	
4 $\div$	5	$e = \frac{D}{r + 3e}$	Let $r=1$ Then $D=0,0719$

Operation $r=1,00$ $0,0719$ ($0,06 = e$
 $+ 3e = ,18$ 708
 Divisor $1,18$ (11) to be rejected.

First $r=1,00$
 $+ e = 0,06$ } $= 1,06 = R$

Then $1 : 0,06 :: 100 : 6$ The Rate per Cent. required.

The First Three Questions may be much more easily perform'd by the following Table, which is only the Amounts of one Pound for Thirty Nine Years.

That

That is, of $R : RR : RRR : R^4 : R^5$. and so on to R^{39} .

Years = 1.	The Amounts of 1l. at 6 per Cent. &c. Compound Interest.	Years = 1.	The Amounts of 1l. at 6 per Cent. &c. Compound Interest.	Years = 1.	The Amounts of 1l. at 6 per Cent. &c. Compound Interest.
1	1,06 = R	14	2,2609039557	27	4,8223459407
2	1,1236 = RR	15	2,3965581931	28	5,1116866971
3	1,191016 = R ³	16	2,5403516847	29	5,4183878990
4	1,26247696	17	2,6927727857	30	5,7434911729
5	1,3382255776	18	2,8543391529	31	6,0881006432
6	1,4185191122	19	3,0255995021	32	6,4533866818
7	1,5036302590	20	3,2071354722	33	6,8405898828
8	1,5938480745	21	3,3995636009	34	7,2510252757
9	1,6894789590	22	3,6035374166	35	7,6860867923
10	1,7908476965	23	3,8197496616	36	8,1472519998
11	1,8982985583	24	4,0489346413	37	8,6360871198
12	2,0121964718	25	4,2918707197	38	9,1542523470
13	2,1329282601	26	4,5493829629	39	9,7035074878

The Title of this Table shews its Construction, and its Use will easily appear by an Example or Two.

Example 1. What will 375l. 10s. Amount to in Nine Years at 6 per Cent. per Annum, &c.

The Tabular Number against 9 Years is 1,689479 which being Multiplied with the Principal 375,5 will produce 634,3993 &c. viz. 634l. 8s. fore, being the Amount or Answer required.

Example 2. What Principal (or Sum) must be put to Interest, to Raise a Stock of 634l. 8s. in Nine Years Time, at 6 per Cent. per Annum, &c.

If the proposed Stock, (viz. 634,4) be Divided by the Tabular Number that is against the given Number of Years (viz. 9) the Quotient will be the Principal (or Sum) required. Viz. Against 9 is 1,689479)

Then 1,689479) 634,4 (375,5 = 375l. 10s. the Principal (or Sum) as was required.

Example 3. In what Time will 375l. 10s. raise a Stock of (or Amount to) 634l. 8s. at 6 per Cent. &c.

Divide

Divide the proposed Stock (*viz.* 634,4) by the given Principal (*viz.* 375,5) and the Quotient will shew the Tabular Number that stands Over-against the Time sought.

Thus 375,5) 634,4 (1,689479 &c. this Number being sought in the Table, will be found to stand against 9 Years, which is the Time required.

But if the Quotient cannot be truly found in the Table of Amounts for Years, as above; Then take out of that Table the nearest Number that is *Less*, and make it a Divisor, by which you must Divide the first Quotient; And then seek the second Quotient in the Table of Amounts for Days, (*which is inserted a little further on*) and it will assign the Number of Days. As in this Example.

In what Time will 563^l. Amount to 860^l. at 6 per Cent. per Annum, Compound Interest?

Answer, In 7 Years and 99 Days.

Thus 563) 860 (1,52753 which shews the Time to be more (*or above*) Seven Years; For over-against 7 Years is 1,50363 which being made the New Divisor:

Viz. 1,50363) 1,52753 (1,01589 &c. this Number is the nearest Amount to 99 Days

Note, If the Stock, Principal, and Time be given; the Rate of Interest will be best found by Extracting the Root, &c. As before in the Fourth Question.

The next thing that I shall here propose, is to make this Table (*which is only Calculated for the Rate of 6 per Cent.*) Universally Useful for all Rates of Compound Interest, *which I may presume to say, is a New Improvement of my own*, being well satisfied it never was Published before, and not only so, but I have heard several very good Artists affirm it was impossible to be done.

The Method of performing it is briefly thus, Let $x =$ the Difference between $1,06 = R$ the Amount of 1^l. for one Year (in the Table) and any other proposed Amount of 1^l. for one Year; which admits of Two Cases.

Case 1. If the proposed Rate be Greater than $1,06 = R$; then will $R + x =$ the true Amount of 1^l. for one Year at that Rate.

Case 2. But if the proposed Rate be Less than $1,06 = R$, then it will be $R - x =$ the Amount of 1/. &c.

Make $\begin{cases} t-1=b . t-2=c . t-3=d . t-4=f \text{ \&c.} \\ \frac{1}{2}tb=g . \frac{1}{3}cg=m . \frac{1}{4}dm=n . \frac{1}{5}fn=s \text{ \&c.} \end{cases}$

Then will $R^t + tR^b x + gR^c xx + mR^d xxx \text{ \&c.} =$ the Amount of 1/. at the given Rate, for any Time denoted by t . In Case 1.

And $R^t - tR^b x + gR^c xx - mR^d xxx \text{ \&c.} =$ the Amount of 1/. in Case 2.

Which is no more but this; Let $R + x$ Or $R - x$ (which soever it is) be Involved (as Directed in Sect. 5. Chap. 2) to the same Power or Height as the Index t the given Time in the Question denotes: Rejecting all the Powers of x above xxx Or $xxxx$ at most, as useless.

Then Multiply that Power of $R + x$ Or $R - x$ into the given Principal, and their Product will be the Amount required. An Example or Two in each Case will render all Easie.

Example 1. Suppose it were required to find what 256l. would Amount to in Fifteen Years, at 8l. per Cent. per Annum Compound Interest? Here $t = 15$.

First $100 : 108 :: 1 : 1,08$ the Amount of 1l. at 8 per Cent.
Next $1,08 - 1,06 = 0,02 = x$. And $R + x = 1,08$ As in Case 1.

Then $R^{15} + 15R^{14}x + 105R^{13}xx + 455R^{12}xxx \text{ \&c.} =$ the Amount of 1l. for 15 Years, at 8 per Cent.

Here $x = 0,02$: $xx = 0,0004$: and $xxx = 0,00008$

By the Table $R^{15} = 2,396558$

And $\begin{cases} 15R^{14}x = 2,260904 \times 15 \times 0,02 = 0,678271 \\ 105R^{13}xx = 2,132928 \times 105 \times 0,0004 = 0,089583 \\ 455R^{12}xxx = 2,012196 \times 455 \times 0,00008 = 0,007324 \end{cases}$

Sum $= 3,171736$

Then $3,171736 \times 256 = 811,964416 =$ A
That is, 811l. 19s. 3½d fere. Which is the Answer as was required.

Example 2. What will 365l. Amount to in Seven Years, at four and a half per Cent, &c.

First $100 : 104,5 :: 1 : 1,045$ the Amount of 1l at 4½ per Cent.

Next

Next $1,06 - 1,045 = 0,015 = x$. Consequently $R - x = 1,045$ as in Case 2.

Then $R^7 - 7R^6x + 21R^5xx - 35R^4xxx$ &c. = the Amount of 1*l*. for 7 Years, at $4\frac{1}{2}$ per Cent.

Here $x = ,015$. $xx = ,000225$. and $xxx = ,000003375$

By the Table $R^7 = + 1,503630$

And $\begin{cases} - 7R^6x & = - 0,148944 \\ + 21R^5xx & = + 0,006323 \\ - 35R^4xxx & = - 0,000141 \end{cases}$

$R^7 - 7R^6x + 21R^5xx - 35R^4xxx = 1,360868$

Then $1,360868 \times 365 = 496,71682 = A$

That is 496*l* . 14*s* . $3\frac{1}{4}$ *d* is the Answer required.

If the Reason of these two Operations be but well understood, it will be very easie to Conceive how to find *P*, the Principal, by having *A*, *t*, and *x* given (because *R* and its Powers are always given by the Table.)

For $R^t \pm tR^bx + gR^cxx \pm mR^dxxx \times P = A$ (as above.)

Therefore $\frac{A}{R^t \pm tR^bx + gR^cxx \pm mR^dxxx} = P$

Or if *A*, *P*, and *t* be given, *x* may be found,

For $R^t \pm tR^bx + gR^cxx \pm mR^dxxx = \frac{A}{P}$. This Equation being Solved, (as in Chap. 10.) the Value of *x* will be found; and then either $R + x$, Or $R - x$ will shew the Rate of Interest, &c.

But I shall leave the Numerical Operations to the Learner's Practice, supposing enough done to shew how all Questions of this kind that are limited by whole Years, may be Computed.

And if the Time given, or sought be not Terminated by whole Years, but by Weeks, Months, Quarters or Half-Years, &c. for Resolving such Questions, the best way will be to Reduce those Parts of a Year into Days; that done, find an Answer according to the Demand of the Question (and agreeing to 1*l*. as before) for those Number of Days; And in order to that, it will be requisite to find the Amount of 1*l*. for one Day, (as in my Compendium of Algebra Page 110.) which I shall here Insert.

Put *a* = the Amount sought, then it will be

$1 : a :: a : aa :: aa : aaa :: aaa : aaaa ::$ to a^{365}

L 1 2

That

That is, $\left\{ \begin{array}{l} \text{As one Pound : Is to its Amount for one Day :: So is} \\ \text{that Amount : To the Amount of Two Days :: And so} \\ \text{is that of Two Days : To that of Three Days. And so} \\ \text{on in } \div \text{ to 365 Days.} \end{array} \right.$

Then the Last of the Terms will be $a^{365} = 1,06$

Put	1	$r + e = a$. And let $r = 1$
1 $\odot$ 365	2	$r^{365} + 365r^{364}e + 66430r^{363}e^2 = a^{365} = 1,06$
2, in Num.	3	$1 + 365e + 66430e^2 = 1,06$
3 — 1	4	$365e + 66430e^2 = 0,06$
4 $\div$ 66430	5	$,00549e + e^2 = 0,0000009032 = D$
5 $\div$	6	$e = \frac{D}{,00549 + e}$

Operation $,00549 \) \ 0,0000009032 \ (,00016 = e$

$+ e = ,00016$

1. Divisor $,0055$

2. Divisor $,00565$

55

3532

3390

First $r = 1$

$+ e = 0,00016$

$r + e = 1,00016$

New $r = 1,00016$ for a Second Operation. Then

2, in Num.	7	$1,06013401407 + 386,887e + 70402,172e^2 = 1,06$ Hence it appears that $r - e = a$
Therefore	8	$1,06013401407 - 386,887e - 70402,172e^2 = 1,06$
8 $\div$	9	$386,887e - 70402,172e^2 = 0,00013401407$
9 $\div$	10	$,0054953 - e^2 = ,0000000019035503$
10 $\div$	11	$e = \frac{,0000000019035503}{,0054953 - e}$

Operation $,0054953 \) \ ,0000000019035503 \ (,00000003 = e$

$- e = ,00000003$

Divisor $,0054950$

164850

255050

219800

$(,00000000464$

Last $r = 1,00016$

$- e = 0,00000003464$

$r - e = a = 1,0001596536$

352503

329700

228030

219800

Which being farther pursued to a Third Operation, it will be $a = 1,000159653587453$ &c.

This

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This Value of a , is the *Amount* of 1*l.* for one Day, from which, if 1*l.* be *Subtracted*, the *Remainder* = ,000159653587 &c. will be the *Interest* of 1*l.* for one Day. Consequently, if any proposed *Principal* be *Multiplied* into either of these, the respective *Product* will be the *Amount*, or *Interest* of that *Principal* for one Day, at 6 per Cent. &c.

And that the *Amount* (or *Interest*) of any *Principal* or *Sum*, may be easily computed for any *Number* of Days *Less* than a Year; I have here inserted the following *Table*, which, with a great deal of Care (and I believe *Exactness*) is Calculated from the Last found (1,000159653587453) *Amount* of 1*l.* for one Day. To which is also *Annexed* a *Table* of the *Amounts* of 1*l.* for *Months*.

Days	Amounts of 1 <i>l.</i> &c.	Days	Amounts of 1 <i>l.</i> &c.	Days	Amounts of 1 <i>l.</i> &c.
1	1,0001596536	26	1,0041592879	51	1,0081749166
2	1,0003193326	27	1,0043196055	52	1,0083358753
3	1,0004790372	28	1,0044799487	53	1,0084968597
4	1,0006387673	29	1,0046403175	54	1,0086578699
5	1,0007985229	30	1,0048007120	55	1,0088189037
6	1,0009583039	31	1,0049611320	56	1,0089799673
7	1,0011181105	32	1,0051215776	57	1,0091410545
8	1,0012779426	33	1,0052820488	58	1,0093021675
9	1,0014378002	34	1,0054425457	59	1,0094633062
10	1,0015976834	35	1,0056030682	60	1,0096244707
11	1,0017575920	36	1,0057636164	61	1,0097856608
12	1,0019175262	37	1,0059241901	62	1,0099468767
13	1,0020774859	38	1,0060847893	63	1,0101081184
14	1,0022374712	39	1,0062454146	64	1,0102693858
15	1,0023974820	40	1,0064060653	65	1,0104306789
16	1,0025575184	41	1,0065667416	66	1,0105919978
17	1,0027175803	42	1,0067274436	67	1,0107533424
18	1,0028776677	43	1,0068881712	68	1,0109147128
19	1,0030377808	44	1,0070489245	69	1,0110761090
20	1,0031979193	45	1,0072097035	70	1,0112375309
21	1,0033580830	46	1,0073705082	71	1,0113989786
22	1,0035182732	47	1,0075313385	72	1,0115604521
23	1,0036784885	48	1,0076921945	73	1,0117219513
24	1,0038387294	49	1,0078530762	74	1,0118834764
25	1,0039989958	50	1,0080139835	75	1,0120450272

Days

Days	Amounts of 1/. &c.	Days	Amounts of 1/. &c.	Days	Amounts of 1/. &c.
76	1,0122066038	116	1,0186908655	156	1,0252166658
77	1,0123682062	117	1,0188535031	157	1,0253803453
78	1,0125298344	118	1,0190161667	158	1,0255440509
79	1,0126914885	119	1,0191788563	159	1,0257077827
80	1,0128531683	120	1,0193415719	160	1,0258715406
81	1,0130148739	121	1,0195043134	161	1,0260353247
82	1,0131766054	122	1,0196670809	162	1,0261991349
83	1,0133383627	123	1,0198298745	163	1,0263629713
84	1,0135001458	124	1,0199926934	164	1,0265268338
85	1,0136619547	125	1,0201555389	165	1,0266907225
86	1,0138237895	126	1,0203184110	166	1,0268546374
87	1,0139856501	127	1,0204813084	167	1,0270185784
88	1,0141475365	128	1,0206442319	168	1,0271825456
89	1,0143094488	129	1,0208071814	169	1,0273465389
90	1,0144713869	130	1,0209701569	170	1,0275105585
91	1,0146333511	131	1,0211331585	171	1,0276746046
92	1,0147953408	132	1,0212961861	172	1,0278386764
93	1,0149573565	133	1,0214592397	173	1,0280027746
94	1,0151193981	134	1,0216223193	174	1,0281668989
95	1,0152814655	135	1,0217854250	175	1,0283310494
96	1,0154435589	136	1,0219485567	176	1,0284952262
97	1,0156056781	137	1,0221117144	177	1,0286594291
98	1,0157678232	138	1,0222748982	178	1,0288236583
99	1,0159299941	139	1,0224381081	179	1,0289879137
100	1,0160921910	140	1,0226013440	180	1,0291521953
101	1,0162544138	141	1,0227646060	181	1,0293165031
102	1,0164166624	142	1,0229278940	182	1,0294808372
103	1,0165789370	143	1,0230912081	183	1,0296451975
104	1,0167412375	144	1,0232545483	184	1,0298095841
105	1,0169035638	145	1,0234179146	185	1,0299739969
106	1,0170659161	146	1,0235813069	186	1,0301384359
107	1,0172282944	147	1,0237447253	187	1,0303029012
108	1,0173906985	148	1,0239081699	188	1,0304673928
109	1,0175513286	149	1,0240716405	189	1,0306319106
110	1,0177155846	150	1,0242351372	190	1,0307964557
111	1,0178780665	151	1,0243986600	191	1,0309610250
112	1,0180405744	152	1,0245622089	192	1,0311256216
113	1,0182031083	153	1,0247257839	193	1,0312902445
114	1,0183656680	154	1,0248893851	194	1,0314548937
115	1,0185282578	155	1,0250530124	195	1,0316195692

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Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.
196	1,0317842709	236	1,0383939484	276	1,0450459680
197	1,0319489990	237	1,0385597318	277	1,0452128133
198	1,0321137534	238	1,0387255415	278	1,0453796853
199	1,0322785341	239	1,0388913778	279	1,0455446584
200	1,0324433410	240	1,0390572405	280	1,0457135092
201	1,0326081742	241	1,0392231298	281	1,0458804611
202	1,0327730339	242	1,0393890454	282	1,0460474397
203	1,0329379198	243	1,0395549876	283	1,0462144449
204	1,0331028321	244	1,0397209563	284	1,0463814768
205	1,0332677706	245	1,0398869515	285	1,0465485353
206	1,0334327355	246	1,0400529732	286	1,0467156206
207	1,0335977268	247	1,0402190214	287	1,0468827325
208	1,0337627444	248	1,0403850961	288	1,0470498711
209	1,0339277883	249	1,0405511973	289	1,0472170363
210	1,0340928586	250	1,0407173250	290	1,0473842283
211	1,0342579552	251	1,0408834793	291	1,0475514469
212	1,0344230782	252	1,0410496601	292	1,0477186923
213	1,0345882275	253	1,0412158674	293	1,0478859643
214	1,0347534033	254	1,0413821012	294	1,0480532631
215	1,0349186054	255	1,0415483616	295	1,0482205885
216	1,0350838338	256	1,0417146485	296	1,0483879407
217	1,0352490887	257	1,0418809620	297	1,0485553196
218	1,0354143699	258	1,0420473021	298	1,0487227252
219	1,0355796775	259	1,0422136687	299	1,0488901576
220	1,0357450115	260	1,0423800618	300	1,0490576166
221	1,0359103719	261	1,0425464815	301	1,0492251025
222	1,0360757587	262	1,0427129278	302	1,0493926150
223	1,0362411719	263	1,0428794007	303	1,0495601543
224	1,0364066116	264	1,0430459001	304	1,0497277204
225	1,0365720776	265	1,0432124261	305	1,0498953132
226	1,0367375701	266	1,0433789787	306	1,0500629327
227	1,0369030889	267	1,0435453579	307	1,0502305790
228	1,0370686342	268	1,0437121637	308	1,0503982521
229	1,0372342059	269	1,0438787961	309	1,0505659519
230	1,0373998041	270	1,0440454551	310	1,0507336786
231	1,0375654287	271	1,0442121407	311	1,0509014320
232	1,0377310798	272	1,0443788529	312	1,0510692121
233	1,0378967573	273	1,0445455918	313	1,0512370191
234	1,0380624612	274	1,0447123572	314	1,0514048529
235	1,0382281916	275	1,0448791493	315	1,0515727134

Days

Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.
316	1,0517406008	339	1,0556094165	362	1,0594924636
317	1,0519085150	340	1,0557779484	363	1,0596616154
318	1,0520764559	341	1,0559465071	364	1,0598307942
319	1,0522444237	342	1,0561150927	365	1,06
320	1,0524124183	343	1,0562837053		
321	1,0525804397	344	1,0564523448		
322	1,0527484880	345	1,0566210112		
323	1,0529165631	346	1,0567897045		
324	1,0530846650	347	1,0569584248		
325	1,0532527937	348	1,0571271720		
326	1,0534209493	349	1,0572959594		
327	1,0535891317	350	1,0574647472		
328	1,0537573410	351	1,0576335753		
329	1,0539255771	352	1,0578024303		
330	1,0540938401	353	1,0579713122		
331	1,0542621300	354	1,0581402211		
332	1,0544304467	355	1,0583091570		
333	1,0545987903	356	1,0584781199		
334	1,0547671608	357	1,0586471097		
335	1,0549355582	358	1,0588161265		
336	1,0551039824	359	1,0589851703		
337	1,0552724336	360	1,0591542411		
338	1,0554409116	361	1,0593233389		
				Months	The Amounts of 1l. at 6 per Cent. For Months.
				1	1,0048675505
				2	1,0097587942
				3	1,0146738462
				4	1,0196128224
				5	1,0245758394
				6	1,0295630141
				7	1,0345744641
				8	1,0396103076
				9	1,0446706634
				10	1,0497556507
				11	1,0548653894
				12	1,06

The Use of this Table is in all respects like that of whole Years, in finding the *Amount* of any given Sum for any proposed Number of Days Less than a Year.

Example 1. Suppose it were required to find the *Amount* of 375l. for 210 Days, at 6 per Cent. &c.

The *Amount* of 1l. for 210 Days is 1,0340928 &c. per Table. Then $1,0340928 \times 375 = 387,7848$ &c. = 387l. 15s. 8d. which is the *Amount* required.

And the rest of the *Variations* may be perform'd just as in the *Examples* of whole Years.

But if the *Time* given, consist of *Years* and parts of a Year As *Quarters*, *Months*, &c. Then Reduce the Odd *Time* of parts of the Year into Days; and the Answer may then be found at Two Operations; As in the following *Example*.

Exa

Example 2. Suppose it were required to find what 265*l.* would Amount to in Five Years and 135 Days, at 6 per Cent. *£c.*

First the Amount of 1*l.* for $\begin{cases} 5 \text{ Years is } 1,338225 \text{ £c.} \\ 135 \text{ Days is } 1,021785 \text{ £c.} \end{cases}$

Then $1,338225 \times 1,021785 \times 265*l.* = 362,355232 \text{ £c.}$ being the Amount or Answer required.

Or, if the Amount and Time are given, To find the Principal; Then Multiply the Amount of 1*l.* for the Years, and the Amount of 1*l.* for the Odd Days together; And by their Product Divide the given Amount, the Quotient will be the Principal required.

Example 3. What Principal will Raise a Stock of 362*l.* .7*s.* 1*½d.* Or 362,355232*l.* in 5 Years and 135 Days, at 6 per Cent. *£c.*

The Amount of 1*l.* for $\begin{cases} 5 \text{ Years is } 1,338225 \text{ £c.} \\ 135 \text{ Days is } 1,021785 \text{ £c.} \end{cases}$

Then $1,338225 \times 1,021785 = 1,367378 \text{ \&c.}$ the Divisor.
Next $1,367378 \mid 362,355232 = A (265*l.*)$ the Principal required.

Again, if the Principal, and its Amount are given, To find the Time, at 6 per Cent. &c. you must Divide the Amount by its Principal, and then proceed as in the Third Example Page 256 for the Answer required.

But if the Amounts and its Principal, with the Time of its being at Interest are given, To find the Rate of Interest; Then proceed as in the Fourth Question Page 255 *£c.*

Now in order to make this Table of Amounts for Days, useful for all Rates of Interest (as before in that for Years) you must First find the Simple Interest of 1*l.* for one Day, both at the given Rate, and also at 6 per Cent. And call their Difference *x*.

Thus, Suppose the given Rate were 8 per Cent. per Annum.
First $100 : 8 :: 1 : 0,08$ And $100 : 6 :: 1 : 0,06$ the Two Simple Interests for one Year.

Then $365 \times 0,08 (0,00021917 \text{ £c.})$ the Simple Interest of 1*l.* for one Day at 8 per Cent.

And $365 \times 0,06 (0,00016438 \text{ £c.})$ the Simple Interest of 1*l.* for one Day at 6 per Cent.

Their Difference $0,00005479 = x$ which may do indifferently well for ordinary small Questions; But where Exactness is required, it will be convenient to make Use of this Proportion:

Viz. { As the Simple Interest of 1*l.* for one Day at 6 per Cent :
Is to the Tabular Interest of 1*l.* for one Day :: So is
the Simple Interest of 1*l.* for one Day, at any given
Rate : To a Fourth Number.

That is, 0,00016438 : 0,00015965 :: 0,00021917 : 0,00021286
Then 0,00021286 — 0,00015965 = 0,00005321 = *x*.

This *x* being Involved with the respective Amounts for Days, in the same manner as was done with those for Years, (vide Page 258) the Result will be the Answer to the Question.

SECT. 2. Annuities, Or Pensions in Arrear, Computed at Compound Interest.

When *Annuities*, &c. are said to be in Arrear, see Page 248. And I shall here make use of the same Letters to represent the same things as before in that Page, save only that *R* is here equal the Amount of 1*l.* as in Section 1. of this Chap.

Suppose *u* = the First Year's Rent of any Annuity without Interest.

Then will $Ru + u =$ { the Amount of the First Year's Rent, and its Interest ; More the 2d Year's Rent,

And $RRu + Ru + u =$ { the Amount of the 1st and 2d Year's Rents, with their Interest ; More the 3d Year's Rent, &c.

Here $RRu + Ru + u = A$ the Amount of any Yearly Rent or Annuity, being forborn Three Years. And from hence may be deduced these Proportions.

Viz. $u : Ru :: Ru : RRu :: RRu : RRRu$ and so on in $\frac{u}{R}$ for any Number of Terms or Years Denoted by t , wherein the Last Term will always be uR^{t-1} .

Consequently $A - uR^{t-1} =$ the Sum of all the Antecedents. And $A - u =$ the Sum of all the Consequents in the Series.

And therefore it will be $u : uR :: A - uR^{t-1} : A - u$
Vide Page 188.

Ergo $Au - uu = RuA - uuR^t$ which being Divided all by u , will become $A - u = RA - uR^t$.

From this Last Equation it will be easie to Raise the following Theorems.

Theorem 1. $\frac{uR^t - u}{R - 1} = A$. **Theorem 2.** $\frac{RA - A}{R^t - 1} = u$

Theo.

Theorem 3. $\left\{ \frac{RA + u - A}{u} = R^t \right.$ If this Equation be continually Divided by R , until nothing Remains, the Number of those Divisions will be t . See Page 255.

4. Again $\left\{ \frac{A}{u} R - R^t = \frac{A - u}{u} \right.$ If this Equation be Resolved into Numbers, according to the Method proposed in Sect. 3. Chap. 10. the Root will shew the Value of R .

Question 1. If 30l. Yearly Rent, Or Annuity, &c. be forborn (viz. remain unpaid) Nine Years; What will it Amount to, at 6 per Cent. per Annum Compound Interest.

Here is given $u = 30$ $t = 9$ And $R = 1,06$ To find A Per Theorem 1.

$R^9 = 1,689479$ By the Table of Amounts for Years

$30 = u$

$R^9 u = 50,684370$

$- u = 30,$

$R - 1 = 0,06$ $20,684370$ ($344,7395 = 344l. 14s. 9\frac{1}{2}d = A$ the Amount required.

Question 2. What Yearly Rent, Or Annuity, &c. being forborn Or unpaid Nine Years, will Raise a Stock of $344l. 14s. 9\frac{1}{2}d = 344,7395$ at 6 per Cent. &c.

Here is given $A = 344,7395$ $t = 9$. And $R = 1,06$ To find u . Per Theorem 2.

$AR = 344,7395 \times 1,06 = 365,42387$

$- A = 344,7395$

$R^t - 1 = 1,689479 - 1 = 0,689479$ $20,68437$ ($30 = u$

Question 3. In what Time will 30l. Yearly Rent, Raise a Stock, Or Amount to $344l. 14s. 9\frac{1}{2}d$ allowing 6 per Cent for the Forbearance of the Payments.

Here is given $u = 30$ $A = 344,7395$ And $R = 1,06$ To find t . Per Theorem 3.

First $AR + u - A = 365,42387 + 30 - 344,7395 = 50,68437$

And $u = 30$ $50,68437$ ($1,689479 = R^t$. Then

$R = 1,06$ $1,689479$ ($1,593848$ And $1,06$ $1,593848$ ($1,50363$ and so on until it become $1,06$ $1,06$ (1 . which will be at the Ninth Division; therefore $t = 9$.

M m 2

Or

Or $R^t = 1,689479$ being sought in the Table of Amounts for Years, will be found to stand Over-against 9 Years, which is the Time required.

Question 4. If 30*l.* per Annum, being unpaid Nine Years, will Amount to 344*l.* 14*s.* 9½*d.* allowing Compound Interest for every Payment as it becomes Due; What must the Rate of Interest be per Cent. &c.

Here is given $u = 30$ $A = 344,7395$ And $t = 9$ To find R by the Last of the Four Equations above, *Viz.* $\left\{ \frac{A}{u} R - R^t = \frac{A-u}{u} \right.$

First $\frac{A}{u} = \frac{344,7395}{30} = 11,491317$ And $\frac{A-u}{u} = 10,491317$

Hence there is this Equation $11,491317R - R^9 = 10,491317$

Let	1	$r - e = R$	And suppose $r = 1$
1 $\odot$ 9	2	$r^9 + 9r^8e + 36r^7ee = R^9$	
1, in Num.	3	$11,491317 - 11,491317e = 11,491317R$	
2, in Num.	4	$1,000000 + 9,000000e + 36ee = R^9$	
3 — 4	5	$10,491317 + 2,491317e - 36ee = 10,491317$	
6 — 1	6	$36ee = 2,491317e$	
6 $\div$ 36e	7	$e = 0,06$ &c.	

First $r = 1, \quad \left. \begin{array}{l} + e = 0,06 \end{array} \right\} = 1,06 = R \quad \left\{ \begin{array}{l} \text{As may be easily Try'd by In-} \\ \text{volving it, and Ordering it,} \\ \text{as the Equation above directs.} \end{array} \right.$

Sect. 3. To find the present Worth of Annuities, Pensions, Or Leases, &c. at Compound Interest.

Let $P =$ the present Worth of any Annuity, or Lease, &c. and the Rest of the Letters as before.

Them, from what has been said in Section 3. Chap. 11. about Purchasing of Annuities, &c. at Simple Interest, it will be Easy to Form the like Theorems here at Compound Interest, *viz.* by Combining Theorem 1. Page 266. And Theorem 1. Page 254. into one Theorem.

For $\left\{ \frac{uR^t - u}{R - 1} = A \right\} \quad \left\{ \begin{array}{l} \text{The Amount of any Yearly Rent being Un-} \\ \text{paid any Number of Years. Per Theo-} \\ \text{rem 1. of the Last Section.} \end{array} \right.$

And $PR^t = A \quad \left\{ \begin{array}{l} \text{The Amount of any Principal or Sum be-} \\ \text{ing put to Interest, for the same Number} \\ \text{of Years. Per Theorem 1. Page 254.} \end{array} \right.$

Hence it follows, That $PR^t = \frac{uR^t - u}{R - 1}$. Viz. $PR^{t+1} - PR^t = uR^t - u$.

being the very same Equation with that in my *Compendium of Algebra* Page 112 which is there Raised from the Consideration of Purchasing Annuities, or Taking of Leases, &c. to be grounded upon a Rank or Series of Geometrical Proportionals Continually Decreasing. Thus $\frac{u}{R}$ is the First and Greatest Term; R the Common Ratio of all the Terms; And P is the Sum of all the Series.

That is, $\frac{u}{R} : \frac{u}{RR} :: \frac{u}{RR} : \frac{u}{RRR} :: \frac{u}{RRR} : \frac{u}{R^4} :: \frac{u}{R^4} : \frac{u}{R^5}$ &c. in ::

until the Last Term $= \frac{u}{R^t}$ Then will $P - \frac{u}{R^t}$ be the Sum

of all the Antecedents. And $P - \frac{u}{R}$ will be the Sum of all the Consequents. Therefore it will be

$\frac{u}{R} : \frac{u}{RR} :: \frac{u}{R} : \frac{u}{R}$ Or (in the same Ratio) $u : \frac{u}{R} :: P - \frac{u}{R^t} : P - \frac{u}{R}$ which produces $PR^{t+1} - uR^t = PR^t - u$. As above.

From this Equation may be Deduced these following Theorems.

Theorem 1. $\left\{ \frac{u - \frac{u}{R^t}}{R - 1} = P \right.$ Theorem 2. $\left\{ \frac{PR^t \times R - PR^t}{R^t - 1} = u \right.$

Theorem 3. $\left\{ \frac{u}{P - u - PR} = R^t \right.$ {Which being Continually divided by R , will give t .

Theorem 4. $\left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^{t+1} \right.$ The Resolving of this Equation, will discover the Value of R .

Question 1. What is 30l. Yearly Rent, to Continue Seven Years, Worth in ready Money; allowing 6 per Cent. Compound Interest to the Purchaser.

Here is given $u = 30$. $t = 7$. And $R = 1.06$ To find P .

Per Theorem 1. Viz. $\frac{u}{R^t} = \frac{30}{1.06^7} = 19.9517$.

And $30 - 19.9517 = 10.0483 = u - \frac{u}{R^t}$

Then

Then $R - 1 = 0,06$ $10,0483$ ($167,4716 = P = 167l. 9s. 5d$ being the Answer required.

Question 2. *What Annuity or Yearly Rent, to Continue Seven Years, may be Purchased for 167l. 9s. 5d. allowing 6 per Cent. Compound Interest to the Purchaser?*

In this Question there is given $P = 167,4716 : t = 7$
And $R = 1,06$ To find u . By the Second Theorem,

$$\text{First } PR^t \times R = 251,8153 \times 1,06 = 266,9242$$

$$\text{And } - PR^t = 167,4716 \times 1,50363 = 251,8153$$

$$\text{Then } R^t - 1 = 0,50363 \quad 15,1089 \quad (30 = u)$$

That is $u = 30l.$ the Answer required.

Question 3. *How Long may One have a Lease of 30l. Yearly Rent, for 167l. 9s. 5d. allowing 6 per Cent. Compound Interest to the Purchaser?*

Here is given $P = 167,4716 . u = 30 .$ And $R = 1,06$ To find t . By the Third Theorem.

$$\text{First } P + u = 167,4716 + 30 = 197,4716$$

$$\text{And } - PR = 177,5199$$

$$\text{Then } 19,9515 \quad 30 = u \quad (1,50363 = R^t)$$

If this $1,50363 = R^t$ be either Continually Divided by $1,06 = R$ until nothing Remain (As before in Page 255.) Or if it be sought in the Table of Amounts for Years, &c. it will discover $t = 7$ which is the true Answer required.

Question 4. *Suppose One should give 167l. 9s. 5d. for the Purchase of a Pension, or Annuity of 30l. per Annum, to Continue Seven Years; At what Rate of Interest, per Cent. would that Purchase be made, allowing Compound Interest to the Purchaser?*

In this Question there is given, $P = 167,4716 . u = 30$ And $t = 7$
To find R . Per Theorem 4.

The 4th Theorem is this Equation $\left\{ \frac{u}{P} = \frac{u}{P} R^t - R^t - R^{t-1} + 1 \right.$

Which being brought into Numbers, and its Root Extracted, as in the 4th Question of the Last Section; the Value of R will be found $1,06$. viz. $R = 1,06$.

And then it will be, $1 : 0,06 :: 100 : 6$ the Rate per Cent. as was required.

The

These Four Questions, Include all the Varieties that can be proposed about Purchasing Annuities or Leases, &c. which are to be either immediately Enter'd upon, Or in Possession at the Time when the Purchase is made.

But such Questions, as relate to Annuities, Or Taking of Leases, &c. in Reversion, must be parted or Divided into Two distinct Questions, each to be separately Consider'd by it self (See Page 252.) As in the following Examples.

Example 1. Suppose it were required to Compute the present Worth of 75^l. Yearly Rent, which is not to Commence or be Enter'd upon, until Ten Years hence; and then to Continue Seven Years after that Time: at 6 per Cent. &c. Compound Interest.

The First Work in this Question, is to find what 75^l. per Annum, to Continue Seven Years, is Worth in ready Money; as if it were to be Immediately enter'd upon: And to perform that, there is given, $u = 75$ $R = 1.06$ And $t = 7$. To find P . As in the First Question of this Section.

Thus, $\frac{u}{R^t} = \frac{75}{1.06^7} = 49,8793$ And $75 - 49,8793 = 25,1207$
 $= u - \frac{u}{R^t}$

Then, $R - 1 = 0.06$ $25,1207 = 418,6783 = 418^l. 13^s. 6^d.
 the Answer to the First Part of the Question.$

Then the next Work will be, to find what Principal or Sum being put out Ten Years, at 6 per Cent. &c. will Amount to 418^l. 13^s. 6^d. Here is given $A = 418,6783$ $R = 1.06$ And $t = 10$. To find P . Per Theorem 2. Page 254.

Thus $R^{10} = 1,790847$ $418,6783 = A$ $(233,7884 = 233^l. 15^s. 9^d.
 the present Worth of 75^l. per Annum in Reversion &c. as was required.$

Example 2. What Annuity Or Yearly Rent to be Entered upon Ten Years hence, and then to Continue Seven Years, may be Purchased for 233^l. 15^s. 9^d. Ready Money, at 6 per Cent. &c. Compound Interest?

In the 1st Work of this Quest. there is given, $P = 233,7884$. $R = 1.06$. And $t = 10$ (the Time which the Annuity is not to Enter'd upon) To find A . Per Theorem 1. Page 254.

Thus, $PR^t = 233,7884 \times 1,790847 = 418,6783 = A$ the Amount

Amount of 233l. 15s. 9d. put to Interest Ten Years, at 6 per Cent. &c. Then for the Second Work of the Question there is given $P = 418,6783$. $R = 1,06$. And $t = 7$ (the Time that the Annuity is to be Enjoy'd) To find u . Per Theorem 2. of this Section.

$$\text{Thus } PR^t \times R = 418,6783 \times 1,50363 \times 1,06 = 667,3095$$

$$- PR^t = 418,6783 \times 1,50363 = 629,5372$$

$$R^t - 1 = 0,50363 \quad 37,7723 (75 = u)$$

That is, $u = 75$ l the Yearly Rent required by the Question.

These Two Examples of finding P and u do fully shew the Method that must be used in Resolving the Two General, and indeed, the most Useful Questions about Annuities, or Leases in Reversion: And if there be Occasion, either the Rate, or the Time, viz. R , Or t , may be found by a due Application of their respective Theorems.

Note, That which hath been done in the Two Last Sections, about Annuities or Yearly Rents, &c. at 6 per Cent. may also be done for any Rate of Interest, by applying the Difference of the Rates (viz. x) As directed in the First Section of this Chapter

Now because that Rents, and Annuities, &c. are usually paid, either by Quarterly, or by Half-yearly Payments, and the Method of Computing them by the Pen, may be thought a little Troublesom; I have Inserted the following Tables of the Amounts of 1l. for Each, at 6 per Cent.

Half-Years 1.	Amounts of 1l. at 6 per Cent. &c. Compound Interest.	Half-Years 1.	Amounts of 1l. at 6 per Cent. &c. Compound Interest.	Half-Years 1.	Amounts of 1l. at 6 per Cent. &c. Compound Interest.
1	1,0215630141	11	1,3777875592	21	1,8437905523
2	1,06	12	1,4185191122	22	1,8982985583
3	1,0913367949	13	1,4604548127	23	1,9544179853
4	1,1236	14	1,5036302590	24	2,0121964718
5	1,1568170026	15	1,5480821017	25	2,0716830644
6	1,191016	16	1,5938480745	26	2,1329282601
7	1,2262260228	17	1,6409670276	27	2,1959840483
8	1,26247696	18	1,6894789589	28	2,2609039557
9	1,2997995842	19	1,7394250493	29	2,3277430911
10	1,3382255776	20	1,7908476965	30	2,3965581931

Quarterly Amounts.

Quarters of a Year = 1	Amounts of 1l. at 6 per Cent. &c. Compound Interest.	Quarters of a Year = 1	Amounts of 1l. at 6 per Cent. &c. Compound Interest.	Quarters of a Year = 1	Amounts of 1l. at 6 per Cent. &c. Compound Interest.
1	1,0146738461	21	1,3578624938	41	1,8171263199
2	1,0295630141	22	1,3777875592	42	1,8437905523
3	1,0446706634	23	1,3980050019	43	1,8708460509
4	1,06	24	1,4185191122	44	1,8982985583
5	1,0755542769	25	1,4393342435	45	1,9261538989
6	1,0913367949	26	1,4604548127	46	1,9544179853
7	1,1073509032	27	1,4818853020	47	1,9830968140
8	1,1236	28	1,5036302590	48	2,0121964718
9	1,1400875335	29	1,5256942978	49	2,0417231330
10	1,1568170026	30	1,5480821017	50	2,0716830644
11	1,1737919574	31	1,5707984203	51	2,1020826228
12	1,191016	32	1,5938480745	52	2,1329282601
13	1,2084927856	33	1,6172359557	53	2,1642265211
14	1,2262260228	34	1,6409670276	54	2,1959840483
15	1,2442194748	35	1,6650463253	55	2,2282075801
16	1,26247696	36	1,6894789589	56	2,2609039557
17	1,2810023527	37	1,7142701133	57	2,2940801123
18	1,2997995842	38	1,7394250493	58	2,3277430912
19	1,3188726433	39	1,7649491048	59	2,3619000349
20	1,3382255776	40	1,7908476965	60	2,3965581931

Either of these Tables may also be made Useful for any proposed Rate of Interest ; by making the $\frac{1}{2}$ Or $\frac{1}{4}$ of the Difference of the Rate = x . &c.

As for Instance, Suppose any of the aforesaid Questions about Annuities or Rents, &c. were to be Computed at 8 per Cent. per Annum.

Then $1,08 - 1,06 = 0,02 = x$ for Yearly Payments ; as before
Consequently 2) $0,02$ ($0,01 = x$. for Half Yearly Payments.

Or 4) $0,02$ ($0,005 = x$. for Quarterly Payments.

Now these Values of x . altho' they are not really true, yet they may serve indifferently well for small Rents ; As I have already said, page 265. But if you would work exactly,

Then $\sqrt{1,08} = 1,0392304845$ &c.

$-\sqrt{1,06} = 1,0295630141$ vide Table Page 272.

Difference = $0,0096674704 = x$ for $\frac{1}{2}$ Yearly Payments.

N n

And

And $\sqrt{\cdot}:\sqrt{1,08} = 1,0194263092 \text{ \&c.}$
 $\text{--- } \sqrt{\cdot}:\sqrt{1,06} = 1,0146738461 \text{ See the Last Table.}$
 Their Difference $0,0047524631 = x. \text{ for Quarterly Payments.}$

These are the true *Values* of x , which being *Involved* with their respective *Amounts* (as before for *Years*, &c.) according as the *Question* requires, the *Result* will be the *Answer* at 8 per Cent, &c. The like may be done for any other *Rate*, either *Greater* or *Less* than 6. Which being a little *Considered* of, will be found very *easie* in *Practice*.

Thus far concerning such *Annuities* or *Leases*, &c. that are *Limited* by any *Assigned Time*; and 'tis only such that can be computed by *Theorems* or certain *Rules*. However it may not perhaps be *Unexceptable*, to insert a brief *Account* of some *Estimates* that have been *Reasonably* made, by *Two* very *Ingenious Persons*, about the *Proportion*, or *Difference* of *Mens Lives*, according to their several *Ages*; which may be of good *Use* in *Computing* the *Values* of *Annuities*, or *taking* of *Leases* for *Lives*, &c.

Sir *William Petty* in his *Discourse*, made before the *Royal Society* (*Anno* 1674) concerning the *Use* of *Duplicate Proportion*, in the *Life* of *Man* and its *Duration*; saith, That its found by *Experience* there are more *Person* *Living* of between 16 and 26 *Years* *Old*, than of any other *Age* Or *Decade* of *Years* in the whole *Life* of *Man* (*viz.* 70 or 80 *Years*). His *Reasons* for that *Assertion* I shall *Omit*; but supposing it *True*, he thence *infers*; That the *Roots* of every *Number* of *Men's Ages* under 16 (whose *Root* is 4) compared with the said *Number* 4, doth shew the *Proportion* of the *Likelihood* of such *Men's* reaching the *Age* of 70 *Years*.

As for *Example*, 'Tis 4 *Times* more *Likely*, that *One* of 16 *Years* *Old* should live to 70, than a *New-born Babe*: 'Tis 3 *Times* more *Likely*, that *One* of 9 *Years* *Old* should attain the *Age* of 70, than the said *Infant*, &c.

On the other hand, 'Tis 5 *To* 4, that *One* of 25 *Years* *Old* will *Die* before *One* of 16: And 6 *To* 5, that *One* of 36 will *Die* before *One* of 25. And so on according to the *Roots* of any other declining *Age*, compared with (4,6) the *Root* of 21, which is the *Year* of *Perfection* according to the *Sense* of our *Law*, and the *Age* for whose *Life* a *Lease* is most *Valuable*.

2. The *Ingenious* and *Great Mathematician*, *Captain Edmund Halley*, (*In Philosoph. Transact. Num.* 196.) doth with great *Industry* and *Skill*, draw an *Estimate* of the *Proportion* of *Men's Lives*

Lives, from the Monthly Tables of the Births and Funerals in Breslaw, the Capital City of the Province of Silesia; Or, as the Germans Call it, Schlesia. Whence he proves, That it's 80 To 1, a Person of 25 Years Old will not Die in a Year: That 'Tis 5½ To 1, That a Man of 40 will Live 7 Years: That a Man of 30 Years Old may reasonably Expect to Live 27 or 28 Years, &c.

Now from these and the like Proportions (he justly Infers, that) the Price of Insurance upon Lives ought to be Regulated, there being a great Difference between the Life of a Man of 20, and One of 50. For Example; 'Tis 100 To 1, that a Man of 20 Dies not in a Year, And but 38 To 1, for a Man of 50 Years of Age. And upon these also depends the Valuation of Annuities for Lives: For it is plain, that the Purchaser ought to Pay only such a Part of the Value of any Annuity, as he hath Chances that he is Living.

And for that Purpose he hath taken the Pains (which was not a little) to Compute the following Table, that shews the Value of Annuities for every Fifth Year of Age to the 70th.

Age	Year's Purchase	Age	Year's Purchase	Age	Year's Purchase
1	10,28	25	12,27	50	9,21
5	13,40	30	11,72	55	8,51
10	13,44	35	11,12	60	7,60
15	13,33	40	10,57	65	6,54
20	12,78	45	9,91	70	5,32

The same Ingenious Gentleman proceeds on, and shews how to Estimate or find the Value of Two Lives, and then of Three Lives, which being too Long a Discourse to be recited here, I have, for Brevities sake, Omitted it; And shall only add this serious Observation.

Viz. How Unjustly we Repine at the Shortness of our Lives, and think our Selves Wrong'd if we attain not to Old Age; whereas it appears, that the One Half of those that are Born, Die in Seventeen Year's Time. For by the aforesaid Bills of Mortality at Breslaw, it was found, that 1238 were in that Time reduced to 616. So that instead of Murmuring at what we Call a short Life, we ought to account it as a great Blessing that we have Survived, perhaps by many Years, that Period of Life whereat the One Half of the Whole Race of Mankind does not Arrive.

Sect. 4. Of Purchasing Free-hold Or Real Estates;
at Compound Interest.

All Free-hold or Real Estates, are supposed to be Purchased or Bought to continue for Ever (viz. without any limited Time); Therefore the Business of Computing the true Value of such Estates, is grounded upon a Rank or Series of Geometrical Proportionals Continually Decreasing ad Infinitum.

Thus, Let P, u, R , Denote the same Data as in the last Section. Then the Series will be, $\frac{u}{R} \cdot \frac{u}{RR} \cdot \frac{u}{R^3} \cdot \frac{u}{R^4} \cdot \frac{u}{R^5}$ and so on in $\therefore$ until the Last Term $= 0$. Then will $P - 0$ (viz. P) be the Sum of all the Antecedents. And $P - \frac{u}{R}$ will be the Sum of all the Consequents; Therefore it will be, $u : \frac{u}{R} :: P : P - \frac{u}{R}$ which produces $PR - u = P$.

This Equation affords these following Theorems.

Theorem 1. $PR - P = u$. Theorem 2. $\left\{ \frac{u}{R - 1} = P \right.$

Theorem 3. $\left\{ \frac{P + u}{P} = R \right.$

Example. Suppose a Free-hold Estate of 75l. Yearly Rent were to be Sold; what is it Worth, allowing the Buyer 6 per Cent. &c. Compound Interest for his Money?

In this Question there is given $u = 75 \cdot R = 1,06$ To find P . Per Theorem 2. Thus $R - 1 = 0,06$ $75 = u$ ($1250l. = P$, the Answer required. And so for any of the rest as Occasion requires. But if the Rent is to be Paid, either by Quarterly, Or Half-Yearly Payments;

Then $R = \sqrt{1,06}$ for Half-Yearly } Payments at 6 per Cent.
And $R = \sqrt[4]{1,06}$ for Quarterly }
Or $\left\{ \begin{array}{l} R = 1,08 \text{ for Yearly} \\ R = \sqrt{1,08} \text{ for Half-Yearly} \\ R = \sqrt[4]{1,08} \text{ for Quarterly} \end{array} \right\}$ Payments at 8 per Cent.

The Like is to be understood for any other proposed Rate of Interest either Greater, Or Less than 6 per Cent.

The Application of these Theorems to Practice, is so very Easie, that its needless to insert more Examples.

A N

INTRODUCTION

TO THE

Mathematicks.

PART III.

CHAP. I.

Of Geometrical Definitions, &c.

Sect. 1. Of Lines and Angles.

I. A Point hath no Parts : That is, a Geometrical Point is not any Quantity, but only an assignable Place in any Quantity denoted by a Point: } *A.* *B.*
As at *A* and *B*.

Such a Place may be conceived so infinitely Small, as to be void of Length, Breadth and Thickness; And therefore a Point may be said to have no Parts.

2. A Line is Called a Quantity of One Dimension, because it may have any supposed Length, but no Breadth nor Thickness, being made, Or represented to the Eye, by the Motion of a Point.

That is, If the Point at *A* be Moved (upon the same Plain) to the Point at *B*, it will describe a Line, either Right, Or Circular, (*viz.* Crooked) according to its Motion.

Therefore the Ends or Limits of a Line are Points.

3. A Right Line, is that Line which Lyeth Ev'n or Streight betwixt those Points that Limit its Length, being the shortest Line that can be drawn between any Two } *A* ————— *B*
Points. As the Line *AB*.

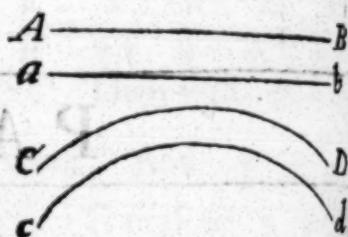
Therefore between any Two Points, there can lie, or be Drawn, but one Right Line.

4. A **Circular, Crooked or Oblique Line**, is that which *Lies* bending between those *Points* which *Limit* its *Length*, As the *Lines* *CD*, Or *FG*, &c.

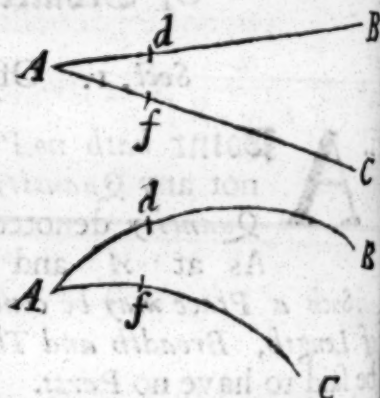
Of these kind of *Lines* there are various *Sorts*; but those of the *Circle*, *Parabola*, *Ellipsis*, and *Hyperbola* are of most general *Use* in *Geometry*; of Which a particular *Account* shall be given further on.



5. **Parallel Lines**, are those that lie equally *Distant* from one another in all their *Parts*, viz. such *Lines* as being *Infinitely* extended (upon the same *Plain*) will never meet; As the *Lines* *AB* and *ab*, Or *CD* and *cd*:



6. **Lines not parallel, but Inclining** (viz. *Leaning*) one towards another, whether they are *Right Lines*, or *Circular Lines*, will (if they are *Extended*) meet, and make an *Angle*; the *Point* where they meet is called the *Angular Point*, As at *A*. And according as such *Lines* stand, nearer or further off each other, the *Angle* is said to be *Lesser* or *Greater*, whether the *Lines* that include the *Angle* be *Long* or *Short*. That is, the *Lines* *Ad*, and *Af* include the same *Angle* as *AB*, and *AC* doth; notwithstanding that *AB* is *Longer* than *Ad*, &c.

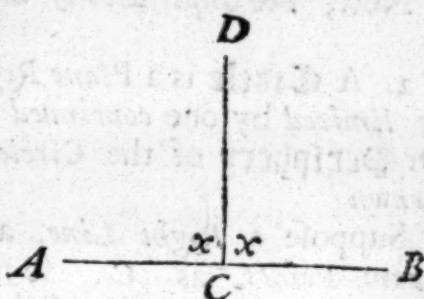


7. All *Angles* included between *Right Lines*, are called *Right-lined Angles*; and those included between *Circular Lines* are called *Spherical Angles*. But all *Angles*, whether *Right-lined* or *Spherical*, fall under one of these *Three Denominations*.

Viz. { A **Right Angle**.
An **Obtuse Angle**.
An **Acute Angle**.

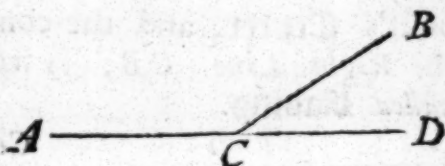
8. A **Right Angle** is that which is included betwixt *Lines* that meet one another *Perpendicular*.

That is, When a *Right-line* as *DC*, meets with another *Right-line* as *AB*, so directly as that it neither *Inclines* nor *Declines* to one *Side* more than the other, but makes the *Angles* on both *Sides* of it equal, as at x, x ; Then are those *Angles* called *Right Angles*; and the *Lines* so meeting are said to be *Perpendicular* to each other.



That is, *AC* and *CB* are *Perpendicular* to *DC*, as well as *DC* is to either or both of them.

9. An *Obtuse Angle* is that which is *Greater* than a *Right Angle*. Such is the *Angle* included between the *Lines AC* and *CB*.



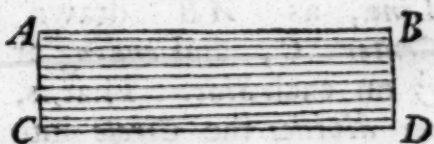
10. An *Acute Angle* is that which is *Less* than a *Right Angle*; As the *Angle* included between the *Lines CB* and *CD*.

These *Two Angles* are generally called *Oblique Angles*.

Sect. 2. Of a Circle, &c.

Before a *Circle* and its *Parts* are *Defined*, it will be convenient to give a *Brief Account* of *Superficies* in general.

1. A *Superficies* or *Surface* is the upper or very *Out-side* of any *visible Thing*. But by *Superficies* in *Geometry*, is meant only so much of the *Out-side* of any *Thing* as is *Inclosed* within a *Line* or *Lines*, according to the *Form* or *Figure* of the *Thing* designed; And it is *produced* or *formed* by the *Motion* of a *Line*, as a *Line* is *described* by the *Motion* of a *Point*; thus: Suppose the *Line AB* were equally *Moved* (upon the same *Plain*) to *CD*; Then will the *Points* at *A* and *B* describe the *Two Lines AC* and *BD*; and by so doing they will form (and inclose) the *Superficies* or *Figure ABCD*, being a *Quantity* of *Two Dimensions*, viz. it hath *Length* and *Breadth*, but not *Thickness*. Consequently the *Bounds* or *Limit* of a *Superficies* are *Lines*.

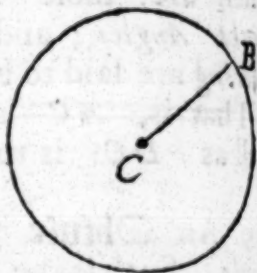


Note,

Note, The Superficies of any Figure is usually Called its Area.

2. A **Circle** is a Plane Regular Figure, whose Area is bounded or limited by one continued Line, called the **Circumference** Or **Periphery** of the Circle, which may be thus described or drawn

Suppose a Right Line, as CB , to have one of it's Extream Points, as C , so fixt upon any Plain, as that the other Point at B may Move about it; Then if the Point at B be Moved Round about, (upon the same Plain) it will describe a Line equally Distant in all it's Parts from the Point C . which will be the Circumference or Periphery of that Circle; the Point C will be it's Center, and the contained Space will be it's Area, And the Right Line CB , by which the Circle is thus described, is called **Radius**.



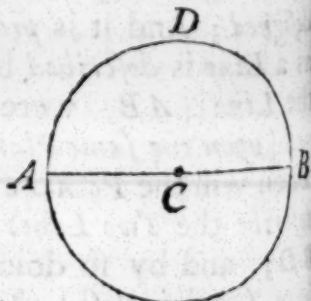
Conseſtary.

From hence 'tis Evident, That an infinite Number of Right Lines may be drawn from the Center of any Circle to Touch it's Periphery, which will be all equal to one another, because they are all Radius's.

And with a little Consideration it will be easie to conceive, that no more than Two equal Right Lines, can be drawn from any Point within a Circle to touch it's Periphery, but from the Center only. (9. e. 3.)

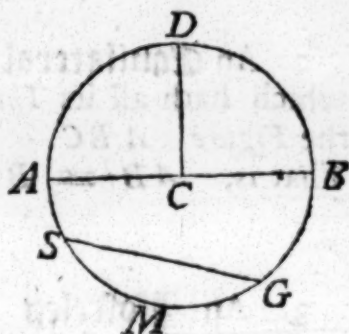
3. **Equal Circles** are those which have equal Radius's; for it's plain by the last Definition, that one and the same Radius (as CB) must needs describe equal Circles, how many soever they are.

4. The **Diameter** of a Circle, is Twice it's Radius joyn'd into one Right Line, as AB drawn through the Center C , and ending at the Periphery on each Side. That is, the Diameter divides the Circle into Two equal Parts.



5. A **Semicircle** (viz. Half a Circle) is a Figure included between the Diameter, and Half the Periphery cut off by the Diameter; As ADB .

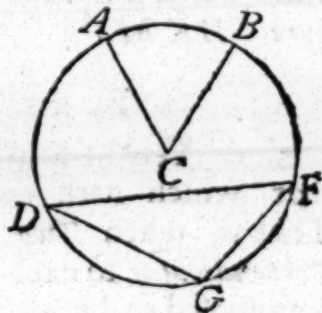
6. A **Quadrant** is Half a Semicircle, viz. one Quarter of a Circle; and it's made by a Radius (as DC) standing Perpendicular upon the Diameter at the Center C , cutting the Periphery of the Semicircle in the middle, as at D . Therefore a Quadrant, or Half the Semicircle is the Measure of a Right Angle.



7. A **Chord Line**, or the Subtense of an Arch, is any Right Line that cuts the Circle into Two unequal Parts, as the Line SG ; and is always Less than the Diameter.

8. A **Segment** of a Circle, is a Figure included betwixt the Chord and that Arch of the Periphery which is cut off by the Chord: And it may either be Greater, or Less than a Semicircle; As the Figure SMG , or SDG .

9. A **Sector** is a Figure included between Two Radius's of the Circle, and that Arch of it's Periphery where they touch, as the Figure ACB . And the Arch AB is the Measure of the Angle at C , included betwixt the Radius's AC and BC .



Note, All Angles of Sectors are called Angles at the Center of a Circle.

10. An **Angle** in the Segment of a Circle, is that which is included between Two Chords that flow from one and the same Point in the Periphery, as at D , and meet with the Ends of another Chord Line, as at F and G :

That is, the Angles at D , at F , and at G , are called Angles at the Periphery, or Angles standing on the Segment of a Circle.

SECT. 3. Of Triangles.

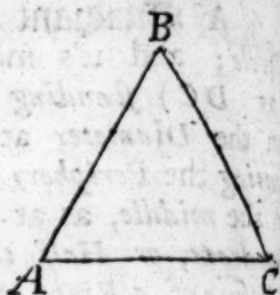
There are Two Kinds of Triangles, viz. Plain and Spherical; but I shall not give any Definition of the Spherical, because they more immediately relate to Astronomy.

1. A **Plain Triangle** is a Figure whose Area is contained within the Limits of Three Right Lines called Sides, including Three Angles: And it may be Divided, and takes its Name either according to its Sides or Angles.

1. By its Sides.

2. An **Equilateral Triangle** is that which hath all its *Three Sides equal*, as the Figure *ABC*.

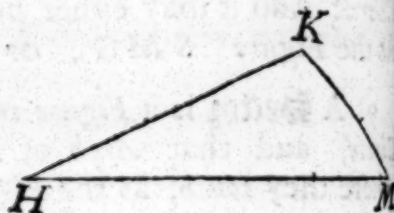
That is, $AB = BC = AC$.



3. An **Isosceles Triangle** is that which hath *only Two* of its *Sides equal*, as the Figure *BDG*. That is, $BD = DG$; but the *Third Side BG* may be either *Greater* or *Less*, as Occasion requires.

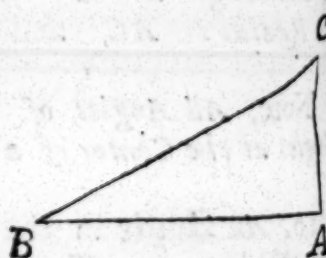


4. A **Scalenois Triangle** is that which hath all its *Three Sides unequal*; such as the Figure *HKM*.



2. By its Angles.

5. A **Right-angled Triangle**, is that which hath *one Right Angle*; That is, when *Two* of its *Sides* are *Perpendicular* to each other, as *CA* is supposed to be, to *BA*. Therefore the Angle at *A*, is a *Right Angle*. Per Def. 8. Sect. 1.



Note, The *Longest Side* of every *Right-angled Triangle* (as *BC*) is called *Hypotenuse*, and the *Longest* of the other *Two Sides* which include the *Right Angle* (as *BA*) is called *Base*. The *Third Side* (as *CA*) is called *Carhetus* or *Perpendicular*.

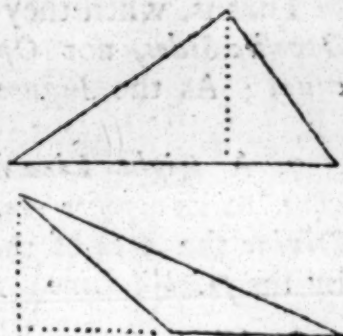
6. An **Obtuse-angled Triangle** is that which hath *one* of its *Angles Obtuse*, and its called an *Amblygonum Triangle*. Such is the *Third Triangle HKM*.

7. An **Acute-angled Triangle** is that which hath all its *Angles Acute*, and its called an *Oxygonum Triangle*; Such are the *First* and *Second Triangles ABC*, and *BDG*.

Note, All *Triangles* that have not a *Right Angle*, whether they are *Acute* Or *Obtuse*, are, in *General Terms*, called *Oblique Triangles*.

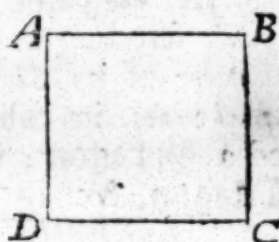
gles, without any other Distinction, as before. And the Longest Side of every Oblique Triangle, is usually called the Base; the other Two are only called Sides, or Legs.

8. The Altitude, or Height of any Plain Triangle, is the Length of a Right Line let fall Perpendicular from any of its Angles, upon the Side opposite to that Angle from whence it falls; And may be either within, or without the Triangle, as Occasion requires, being denoted by the Two prick'd Lines in the Annexed Triangles.



SECT. 4. Of Four-Sided Figures, &c.

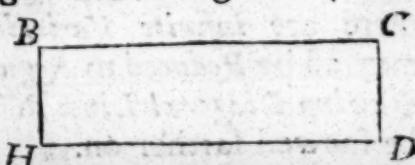
1. A Square, is a plain regular Figure, whose Area is Limited by Four equal Sides all Perpendicular one to another. That is, when $AB = BC = CD = DA$. and the Angles A, B, C, D are all equal, Then its usually called a Geometrical Square.



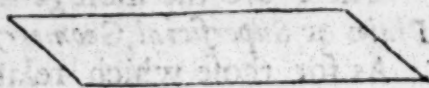
2. A Rhombus or Diamond-like Figure, is that which hath Four equal Sides, but no Right Angle. That is, a Rhombus is a Square Moved out of its Right Position, as the Annexed Figure.



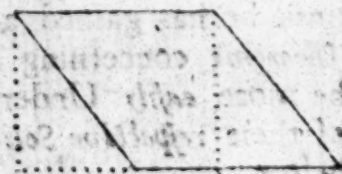
3. A Rectangle, Or a Right-angled Parallelogram (often called an Oblong, or Long Square) is a Figure that hath Four Right Angles, and its Two opposite Sides equal, viz. $BC = HD$ and $BH = CD$.



4. A Rhomboides, is an Oblique-angled Parallelogram. That is, it is a Parallelogram Moved out of its Right Position, like the Annexed Figure.



5. The Altitude or Height of any Oblique-angled Parallelogram, viz. either of the Rhombus, or Rhomboides, is a Right Line let fall Perpendicular from any Angle upon the Side Opposite to that Angle; and it may either be within or without the Figure. As the prick'd Lines in the Annexed Figure.



6. All *Four Sided Figures*, which differ from those before mention'd, are called *Trapezias*.

That is, when they have neither *Opposite Sides*, nor *Opposite Angles* equal; As the Figure *ABCD*.



7. A *Right Line* drawn from any *Angle* in a *Four Sided Figure*, to its *opposite Angle*, is called a *Diagonal Line*, and will *Divide* the *Area* of the *Figure* into *Two Triangles*, being denoted by the prick'd Line *AC* in the last Figure.

8. All *Right-lined Figures* that have more than *Four Sides*, are called *Polygons*, whether they be *Regular*, or *Irregular*.

9. A *Regular Polygon*, is that which hath all its *Sides* equal, standing at equal *Angles*; And is nam'd according to the *Number* of its *Sides* (or *Angles*): That is, if it have *Five* equal *Sides*, its called a *Pentagon*; if *Six* equal *Sides*, its called a *Hexagon*; if *Seven*, it is a *Heptagon*; if *Eight*, its an *Octagon*, &c.

Note, All *Regular Polygons* may be inscribed in a *Circle*. That is, their *Angular Points*, how many soever they have, will all just touch the *Circle's Periphery*.

10. An *Irregular Polygon* is that *Figure* which hath many *unequal Sides* standing at *unequal Angles* (like unto the *Annexed Figure*, or otherwise); And of such kind of *Polygons* there are *infinite Varieties*; But they may all be *Reduced* to *Regular Figures*, by drawing *Diagonal Lines* in them, as shall be shewed farther on.



These are the most *general* and *useful Definitions* that concern *Plain* or *Superficial Geometry*.

As for those which relate to *Solids*, I thought it convenient to omit giving any *Account* of them in this *Place*, because they would rather *Puzzle* and *Amuse* the *Learner* than *Improve* him, until he has gained a *competent Knowledge* in the most *Useful Theorems* concerning *Superficies*, for then those *Definitions* may be more *easily Understood*, and will help to form a *clearer Idea* of their *respective Solids*, than 'tis possible to conceive of them before.

Sect. 5. Of such Terms as are generally used in Geometry.

Whatsoever is proposed in Geometry, will either be a **Problem**, or a **Theorem**.

Both which Euclid includes in the general Term of Proposition.

A **Problem** is that which proposes something to be done, and Relates more Immediately to Practical than Speculative Geometry; That is, its generally of such a Nature, as to be perform'd by some known or common received Rules, without any regard had to their Inventions or Demonstrations.

A **Theorem** is when any common received Rule, or any New Proposition is required to be Demonstrated, that so it may from thenceforward become a Certain Rule to be Relied upon in Practice, when Occasion requires it. And therefore several Rules are often called Theorems, by which Operations in Arithmetick, and Conclusions in Geometry are perform'd.

Note, By **Demonstration** is understood the highest Degree of Proof that human Reason is capable of attaining to, by a Train of Arguments, deduced or drawn from such plain Axioms, and other Self-evident Truths, as cannot be denied by any One that considers them.

A **Corollary**, Or **Consectary**, is some Consequent Truth drawn, or gained from any Demonstration.

A **Lemma** is the Demonstration of some Premiss laid down, or proposed as a Preparative, to obviate and shorten the Proof of the Theorem under Consideration.

A **Scholium** is a brief Commentary, or Observation made upon some precedent Discourse.

N. B. I advise the young Geometer to be very perfect in the Definitions; viz. not to rest satisfied with a bare Remembrance of them, but that he endeavour to gain a clear Idea or Understanding of the Things defined. And for that Reason I have been fuller in every Definition than is usual.

C H A P. II.

The First Rudiments, or Leading and Preparatory Problems in Plain Geometry.

In order to perform the following Problems, the young Geometer ought to be provided with a Thin Streight Ruler, made either of Brass, or Box-wood; And Two Pair of very good Compasses, viz. one Pair called Three Pointed Compasses, being very Useful for drawing of Figures or Schemes, either with black Lead or Ink, And one Pair of plain Compasses with very fine Points, to Measure and set off Distances; Also he should have a very good Steel drawing Pen: And then he may proceed to the Work with this Caution; That he ought to make himself Master of one Problem before he undertake the next. That is, he ought to understand the Design, and, as far as he can, the Reason of every Problem as well as how to do it, and then a little Practice will render them very Easie, they being all grounded upon these following Postulates.

Postulates or Petitions.

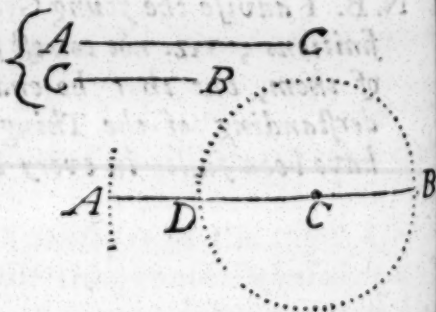
1. That a Right Line may be drawn from any one given Point to another.
2. That a Right Line may be Produced, Increased, or made Longer from either of its Ends.
3. That upon any given Point (or Center) and with any given Distance (viz. with any Radius) a Circle may be described.

P R O B L E M I.

Two Right Lines being given; To find their Sum and Difference. (3. e. 1.)

Let the given Lines be

Make the shortest Line, as CB, Radius, and with it describe a Circle; From its Center C set off the other Line AC, and joyn ACB with a Right Line. Then will $AB = AC + CB$; and $AD = AC - CB$; as was required.

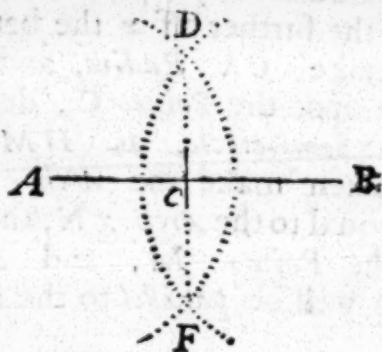


P R O

PROBLEM II.

To Biseſt, Or Divide a Right Line given (as AB) into Two Equal Parts. (10. e. 1.)

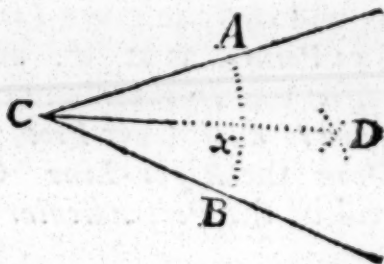
From both Ends of the given Line (viz. A and B) with any Radius Greater than Half its Length, deſcribe Two Arches, that may croſs each other in Two Points, as at D and F ; Then joyn thoſe Points D , F with a Right Line; And it will biſeſt the Line AB in the Middle at C ; viz. it will make $AC = CB$; as was required.



PROBLEM III.

To Biſeſt a Right-Lin'd Angle given, into Two equal Angles. (9. e. 1.)

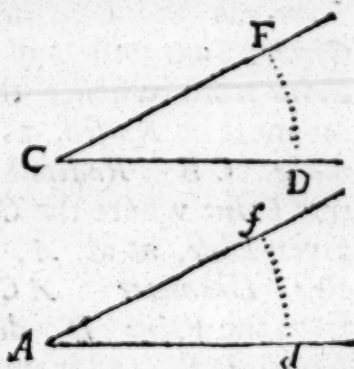
Upon the Angle Point, as at C , with any convenient Radius, deſcribe an Arch as AB ; And from thoſe Points A and B , deſcribe Two equal Arches croſſing each other, as at D . Then joyn the Points C , and D , with a Right Line, and it will biſeſt the Arch AB , and conſequently the Angle, as was required.



PROBLEM IV.

At a Point A , in a Right Line given AB , To make a Right-Lin'd Angle, equal to a Right-Lin'd Angle given C . (23. e. 1.)

Upon the given Angle Point C deſcribe an Arch, as FD (making CD any Radius at pleaſure) and with the ſame Radius, deſcribe the like Arch upon the given Point A , as fd . That is, make the Arch fd Equal to the Arch FD ; Then joyn the Points A , and f with a Right Line, and it will Form the Angle required.

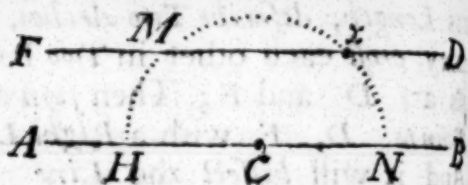


PRO-

PROBLEM V.

To draw a Right Line, as FD , parallel to a given Right Line AB , That shall pass through any assigned Point, as at x , viz. at any Distance required. (31. e. 1.)

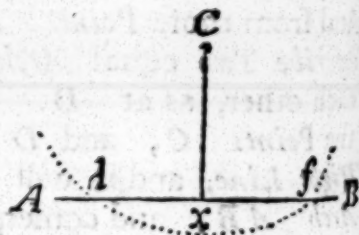
Take any convenient Point in the given Line, as at C , (the further off x the better); make Cx Radius, and with it upon the Point C , describe a Semi-circle, as $HMxN$. Then make the Arch HM equal to the Arch xN ; through the Points M , and x , draw the Right Line FD , and it will be parallel to the Line AC , as was required.



PROBLEM VI.

To let fall a Perpendicular, as Cx , upon a given Right Line AB , from any assigned Point that is not in it, as from C . (12. e. 1.)

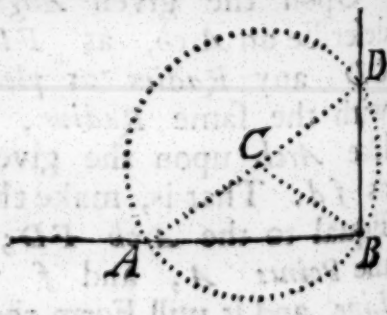
Upon the given Point C , describe such an Arch of a Circle, as will cross the given Line AB in Two Points, as at d , and f ; Then bisect the Distance between those Two Points d, f (per Prob. 2.) as at x . Draw the Right Line Cx , and it will be the Perpendicular required.



PROBLEM VII.

To Erect Or Raise a Perpendicular upon the End of any given Right Line, as at B ; Or upon any other Point assigned in it. (11. e. 1.)

Upon any Point (taken at an Adventure) out of the given Line, as at C , describe such a Circle, as will pass through the Point from whence the Perpendicular must be Raised, as at B (viz. make CB Radius); And from the Point where the Circle cuts the given Line, as at A , draw the Circle's Diameter ACD . Then from the Point D , draw the Right Line DB , and it will be the Perpendicular as was required.

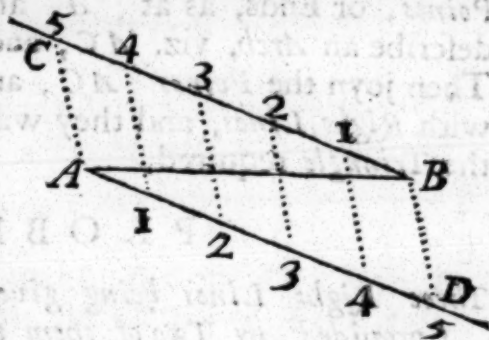


P R O

PROBLEM VIII.

To Divide any given Right Line, as AB , into any proposed Number of Equal Parts. (10. e. 6.)

At the Extream Points (or Ends) of the given Line, as at A and B , make Two equal Angles (by Prob. 4.) continuing their Sides AD , and BC to any sufficient Length; Then upon those Sides, beginning at the Points A , and B , set off the proposed Number of equal Parts (suppose them 5.) If Right Lines be drawn (cross the given Line) from one Point to the other, as in the Annexed Figure, those Lines will Divide the given Line AB into the Number of equal Parts required.

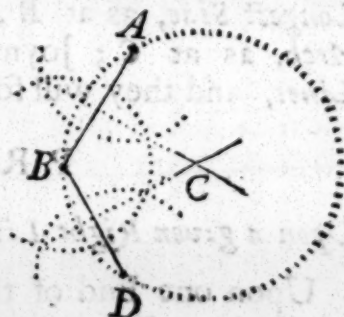


PROBLEM IX.

To describe a Circle that shall pass (or cut) through any Three Points given, not lying in a Right Line, as the Points A , B , D .

Join the Points BA , and BD with Right Lines, then Bisect both those Lines (per Problem 2.) The Point where the bisecting Lines meet, as at C , will be the Center of the Circle required.

The Work of this Problem being well understood, it will be Easie to perform the Two following; without any Scheme.



viz .1. To find the Center of any Circle given. (1. e. 3.)

By the Last Prob. 'tis plain, that if Three Points be any where taken in the given Circle's Periphery, as at A , B , D , the Center of that Circle may be found as before.

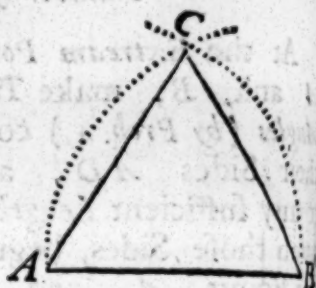
2. If a Segment of any Circle be given; To compleat or describe the whole Circle. (23. e. 3.)

This may be done by taking any Three Points in the given Segment's Arch, and then proceed as before.

PROBLEM X.

Upon a Right Line given, as AB ; To describe an Equilateral Triangle. (1. e. 1.)

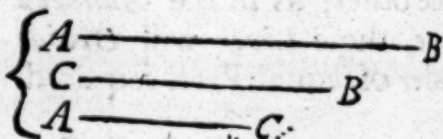
Make the given Line Radius, and with it, upon each of its Extream Points, or Ends, as at A , and B , describe an Arch, viz. AC , and BC . Then joyn the Points AC , and BC with Right Lines, and they will make the Triangle required.



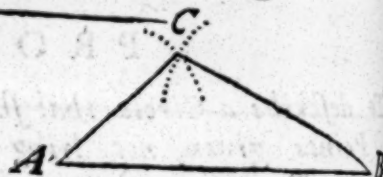
PROBLEM XI.

Three Right Lines being given; To form them into a Triangle (provided any Two of them taken together be Longer than the Third. (22. e. 1.)

Let the given Lines be



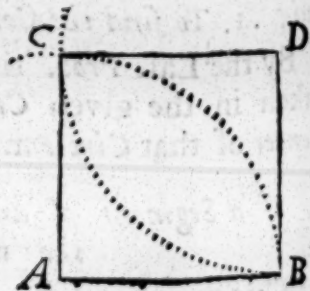
Make either of the shorter Lines, as AC , Radius, and upon either End of the Longest Line, as at A , describe an Arch; then make the other Line CB Radius, and upon the other End of the Longest Side, as at B , describe another Arch to cross the First Arch, as at C ; Joyn the Points CA , and CB with Right Lines, and they will form the Triangle required.



PROBLEM XII.

Upon a given Right Line, as AB ; To form a Square. (46. e. 1.)

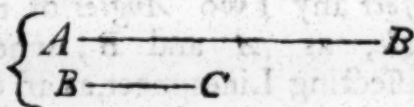
Upon one End of the given Line, as at B , Erect the Perpendicular BD , equal in Length with the given Line, viz. make $BD = AB$; that being done, make the given Line Radius, and upon the Points A , and D , describe equal Arches to cross each other, as at C ; Then joyn the Points CA , and CD with Right Lines, and they will form the Square required.



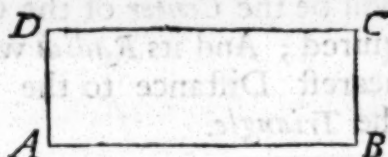
PROBLEM XIII.

Two unequal Right Lines being given ; To form or make of them a Right-angled Parallelogram.

Let the given Lines be



Upon one End of the Longest Line, as at B, Erect a Perpendicular, of the same Length with the shortest Line BC ; Then from the Point C draw a Line Parallel, and of the same Length to AB, viz. make $DC = AB$. Joyn DA with a Right Line, and it will form the Oblong or Parallelogram required.



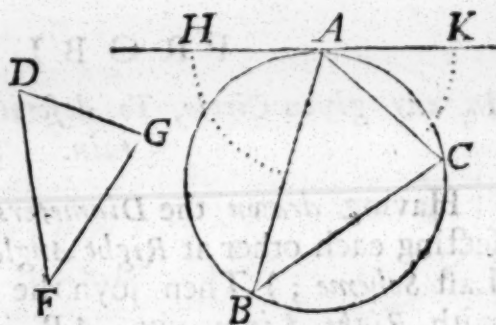
As for Rhombus's, and Rhomboides, to wit, Oblique-angled Parallelograms, they are made or described after the same manner with the Two last Figures ; only instead of Erecting the Perpendiculars, you must set off their given Angles, and then proceed to draw their Sides Parallel, &c. As before.

PROBLEM XIV.

In any given Circle, To Inscribe or Make a Triangle, whose Angles shall be equal to the Angles of a given Triangle, as the Triangle FDG. (2. e. 4.)

Note, Any Right-lin'd Figure is said to be Inscribed in a Circle, when all the Angular Points of that Figure do just touch the Circle's Periphery.

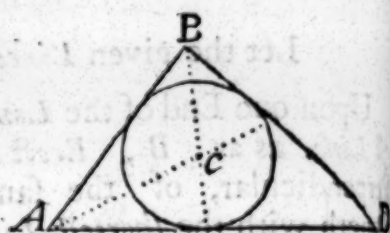
Draw any Right Line (as HK) so as to just touch the Circle, as at A ; Then make the Angle KAC equal to any one Angle of the given Triangle (as DFK) ; And the Angle HAB equal to another Angle of the Triangle (as DGF) ; Then will the Angle BAC be equal to the Angle FDG. Joyn the Points B and C with a Right Line, and it will form the Triangle required,



PROBLEM XV.

In any given Triangle, as ABD , To describe a Circle that shall touch all its Sides. (4. e. 4.)

Bisect any Two Angles of the Triangle, as A and B , where the bisecting Lines meet, as at C , will be the Center of the Circle required; And its Radius will be the nearest Distance to the Sides of the Triangle.



PROBLEM XVI.

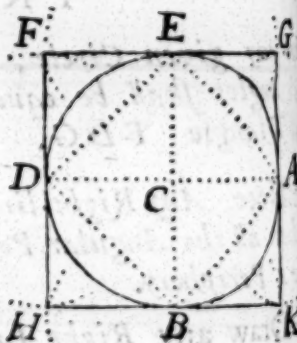
To describe a Circle about any given Triangle. (5. e. 4.)

This Problem is perform'd in all Respects like the 9th. viz. by Bisecting any Two Sides of the given Triangle; the Point where those bisecting Lines meet, will be the Center of the Circle required.

PROBLEM XVII.

To describe a Square about any given Circle. (7. e. 4.)

Draw Two Diameters in the given Circle, As DA , and EB , at Right Angles in the Center C ; And with the Circle's Radius CA , describe from the Extream Points of the Diameters A, B, D, E , Cross Arches, as at F, G, H, K ; Then joyn those Points where the Arches Cross, with Right Lines, and they will form the Square required.



PROBLEM XVIII.

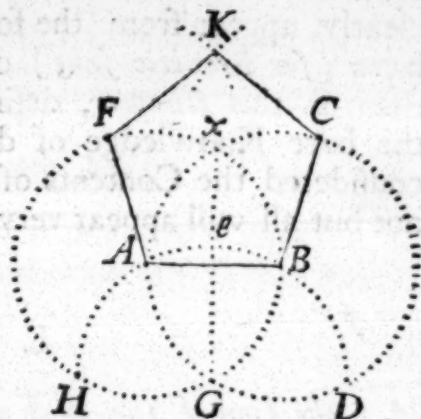
In any given Circle, To describe the Largest Square it can contain. (6. e. 4.)

Having drawn the Diameters, as DA , and EB , Bisecting each other at Right Angles in the Center C , (As in the Last Scheme;) Then joyn the Points A, B, D , and with Right Lines, viz. AB, BD, DE, EA , and they will be the Sides of the Square required.

PROBLEM XIX.

Upon any given Right Line, as AB , To describe a Regular Pentagon or Five Sided Polygon.

Make the given Line Radius, and upon each End of it, describe a Circle, and through those Points where the Circles cross each other, as at G , x , draw the Right Line Gex upon the Point G , with the same Radius describe the Arch $HaeBD$. Then lay a Ruler upon the Points De , and mark where it crosses the other Circle, as at F . Again, lay the Ruler upon the Points He , and mark where it crosses the other Circle, as at C . Then from the Points F and C (with the same Radius as before) describe cross Arches, as at K . Joyn the Points AF , FK , KC , and CB with Right Lines, and they will form the Pentagon required.



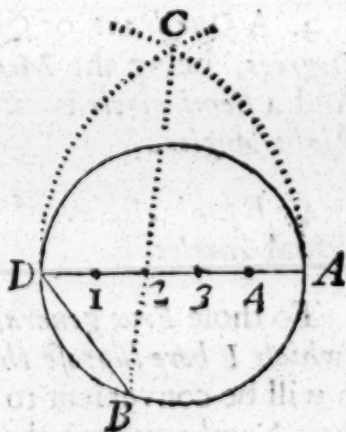
Viz. $AF = FK = KC = CB = AB$. And the Angles at A , B , C , K , F will be equal.

PROBLEM XX.

In any given Circle, To describe a Regular Pentagon. (II. e. 4. and 10. e. 3.)

Or, in general Terms, to describe any Regular Polygon in a Circle.

Draw the Circle's Diameter DA , and divide it into so many equal Parts, as the proposed Polygon hath Number of Sides; Then make the whole Diameter a Radius, and describe the Two Arches CA , and CD . If a Right Line be drawn from the Point C , through the Second of those equal Parts in the Diameter, as at 2, it will assign a Point in the opposite Semicircle's Periphery, as at B . Joyn DB with a Right Line, and it will be the true Side of the Pentagon required, &c.



These

These Twenty Problems are sufficient to Exercise the young Practitioner, and bring his Hand to the right Management of a Ruler and Compasses, wherein I would advise him to be very Ready and Exact.

As to the Reason why such Lines must be so drawn, as directed at each Problem; That, I presume, will fully and clearly appear from the following Theorems; and therefore I have (for Brevities sake) omitted giving any Demonstrations of them in this Chapter, desiring the Learner to be satisfied with the bare Knowledge of doing them only, until he hath fully considered the Contents of the next Chapter; and then I doubt not but all will appear very Plain and Easy.

C H A P. III.

A Collection of the most useful Theorems in plain Geometry Demonstrated.

Note, In order to shorten several of the following Demonstrations, it will be necessary to premise, That

1. The Periphery (or Circumference) of every Circle (whether Great or Small) is supposed to be Divided into 360 equal Parts, called Degrees; And every one of those Degrees are Divided into 60 equal Parts called Minutes, &c.
2. All Angles are Measured by the Arch of a Circle described upon the Angular Point (see Defi. 9. Page 281.) and are esteemed Greater or Less, according to the Number of Degrees contained in that Arch.
3. A Quadrant or Quarter Part of any Circle, is always 90 Degrees, being the Measure of a Right Angle (Defi. 6. Page 281.) And a Semi-circle is = 180 Degrees, being the Measure of Two Right Angles.
4. Equal Arches of a Circle; Or of Equal Circles, Measure Equal Angles.

To those Five general Axioms, already laid down in Page 146 (which I here suppose the Reader to be very well acquainted with) it will be convenient to understand these following, which begin their Number where the other ended.

Axioms.

AXIOMS.

6. Every whole Thing is Greater than its part.

That is, the whole Line AB is }
Greater than its Part Ac , &c. } $A \text{ --- } | \text{ --- } B$
The same is to be understood of Superficies and Solids.

7. Every Whole is Equal to all its Parts taken together.

That is, the whole Line AB is equal }
to its Parts $AC + cd + de + eB$. } $A \text{ --- } | \text{ --- } | \text{ --- } B$
The same is also True in Superficies and Solids.

8. Those Things which being Laid one upon another, do agree Or meet in all their parts, are equal one to the other.

But the Converse of this Axiom; to wit, that equal things being laid one upon the other will meet, is only true in Lines and Angles, but not in Superficies, unless they be alike, viz. of the same Figure or Form; As for Instance, a Circle may be equal in Area to a Square; but if they are Laid one upon the other, 'tis plain they cannot meet in all their Parts, because they are unlike Figures. Also a Parallelogram, and a Triangle may be Equal in their Area's one to another; And both of them may be Equal to a Square; But if they are Laid one upon the other, they will not meet in all their Parts, &c.

Note, Besides the Characters already explained in Part 1. And in other Places of this Tract, these following are Added.

Viz. $\angle$ denotes an Angle in general, and $\angle \angle$ signifies Angles; $\triangle$ signifies a Triangle; $\square$ signifies a Square; and $\square$ denotes a Parallelogram. And when an Angle is Denoted by any Three Letters (as ABC , &c.) the middle Letter (As B) always Denotes the Angular Point; and the other Two Letters (As AB , and BC) Denotes the Lines, or Sides of a Triangle which include that Angle.

These things being premised, the young Geometer may proceed to the Demonstrations of the following Theorems; wherein he may perceive an absolute Necessity of being well versed in several things that have been already delivered: And also it will be very Advantageous to store up several useful Corollaries, and Lemma's as they become discovered Truths; For it often happens, that a Proposition cannot be clearly Demonstrated a priori, or of it Self, without a great deal of Trouble. Therefore it will be Useful to have Recourse to those Truths that may be assisting in the Demonstration then in hand.

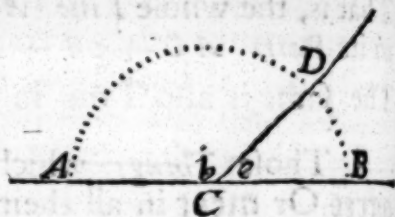
THE O.

THEOREM I.

If a Right Line stand upon (or meet with) another Right Line, and make Angles with it, they will either be Two Right Angles, or Two Angles equal to Two Right Angles. (13. e. 1.)

Demonstration.

Suppose the Lines to be AB and DC ; meeting in the Point at C ; upon C describe any Circle at pleasure. Then will the Arch AD be the Measure of the $\angle b$, and the Arch DB the Measure of the $\angle e$; but the Arches $AD + DB = 180^\circ$. *Viz.* they complet the Semicircle.



Consequently, the $\angle b + \angle e = 180^\circ$. Which was to be proved.

Corollaries.

1. Hence it follows, that if the $\angle b = 90^\circ$. then $\angle e = 90^\circ$; but if $\angle b$ be Obtuse, then the $\angle e$ will be Acute, &c.

2. From hence it will be easie to Conceive, that if several Right Lines stand upon, or meet with any Right Line, at one and the same Point, all the Angles taken together will be $= 180^\circ$. *viz.* Two Right Angles.

THEOREM II.

If Two Right Lines Intersect (*viz.* cut or cross) each other, the Two opposite Angles will be Equal. (15. e. 1.)

Demonstration.

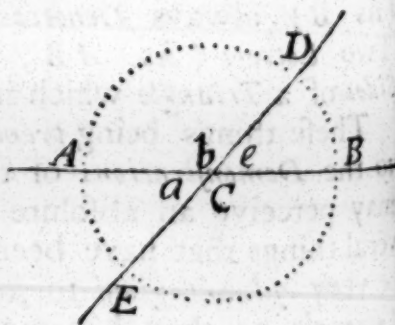
Let the Two Lines be AB , and DE , Intersecting each other in the Center C .

Then $\angle b + \angle e = 180^\circ$.
And $\angle b + \angle a = 180^\circ$. } *per Last.*

Consequently $\angle b + \angle e = \angle b + \angle a$
per Axiom. 5.

Subtract $\angle b$ on both Sides of the Equation; and it will leave $\angle e = \angle a$.

Again $\angle b + \angle e = 180^\circ$. as before; and $\angle e + \angle c = 180^\circ$.
Consequently $\angle e + \angle c = \angle b + \angle e$. Subtract $\angle e$ and then $\angle c = \angle b$. **Q. E. D.**



Corollary

Corollary.

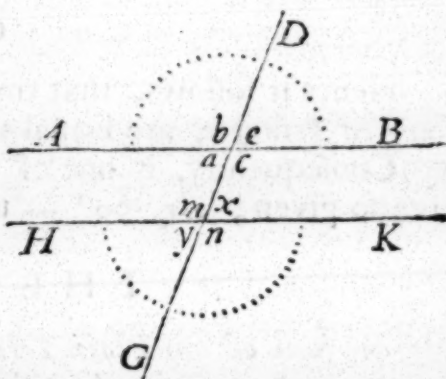
From hence its *Evident*, that if Two Lines Intersect each other, they will make Four *Angles*, which being taken together, will always be Equal to Four *Right Angles*.

THEOREM III.

If a *Right Line* cut (or cross) Two *Parallel Lines*, it will make the *Opposite Angles* equal one to another. (29. e. 1.)

Demonstration.

Suppose the Two Lines *AB*, and *HK* to be *Parallel*, and the *Right Line* *DG*, to cut them both, at *C* and *n*. Upon the Point *C* (with any *Radius*) describe a *Semicircle*, and with the same *Radius*, upon the Point at *n*, describe another *Semicircle* opposite to the First, as in the Figure. Then 'tis plain, and I suppose, very *Easie* to conceive, that if the Center *C* were *Moved* along upon the Line *DG*, until it came to the Center at *n*, the Two Lines *AB*, and *HK* would meet and concur, viz. become one Line (for *Parallel Lines* are as it were but one broad Line.) Consequently the Two *Semicircles* would also meet, and become one entire *Circle*, like to that in the Last *Demonstration*. And therefore the $\angle y = \angle x = \angle a = \angle e$ } {as before, per
And $\angle m = \angle n = \angle b = \angle c$ } {Last Theorem.
Q. E. D.



Corollary.

Hence it follows, that if *Three*, *Four*, or never so many *Parallel Lines* are cut or cross'd by one *Right Line*, all their opposite *Angles* will be Equal.

THEOREM IV.

The *Three Angles* of every *Plain Triangle*, are Equal to Two *Right Angles*. (32. e. 1.)

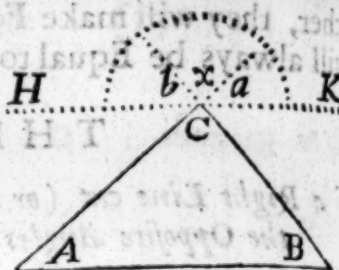
Consequently any Two *Angles* of any *Plain Triangle*, must needs be Less than Two *Right Angles*. (17. e. 1.)

Q q

Demon-

Demonstration.

Let the $\triangle ABC$ be proposed; draw the *Right Line* HK , Parallel to the Side AB , just touching the *Vertical Angle* C ; And upon the same *Angular Point* C , describe any *Semicircle*, and produce the Sides AC , and BC to its *Periphery*. Then will $\angle b = \angle B$, $\angle a = \angle A$, and $\angle x = \angle C$, Per *Last Theorem*.



But $\angle b + \angle a + \angle x = 180^\circ$. or Two *Right Angles*. Consequently $\angle B + \angle A + \angle C = 180^\circ$. Per *Axiom 5*. Q. E. D.

Corollary.

Hence it follows, that the Two *Acute Angles* of every *Right-angled Triangle*, are Equal to a *Right Angle* or 90° .

Consequently, if one of the *Acute Angles* be given, the other is also given; viz. 90° — the given $\angle$ leaves the other $\angle$.

THEOREM V.

If one Side of any Plain Triangle, be continued or produced beyond, or out of the Triangle; the outward Angle will always be Equal to the Two inward Opposite Angles. (32. e. 1.)

Demonstration.

Let the Side AB of the $\triangle ABC$ be produced out of the $\triangle$, suppose to D , &c. as in the Figure. Then $\angle z = \angle A + \angle C$, for the $\angle B + \angle z = 180^\circ$. Per *Theorem 1*.



And the $\angle B + \angle A + \angle C = 180^\circ$. Per *Last Theorem*. Therefore $\angle B + \angle z = \angle B + \angle A + \angle C$. Per *Axiom 1*. Subtract $\angle B$ on both Sides the Equation, and it will leave $\angle z = \angle A + \angle C$. (Per *Axiom 2*.) Q. E. D.

Consequently, the outward Angle (at z) of any Plain Triangle must needs be greater than either of the inward opposite Angles viz. A or C . (16. e. 1.)

Corollary.

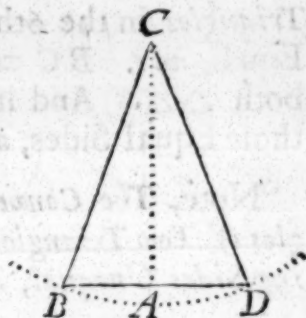
Hence it follows, That if One Angle of any Plain Triangle be given, the Sum of the other Two Angles is also given, 180° — the given $\angle$ = the other Two $\angle$'s.

THEOREM VI.

In every Plain Triangle, Equal Sides subtend (viz. are Opposite to) Equal Angles. (5. e. 1.)
Consequently, Equal Angles are subtended by Equal Sides. (6. e. 1.)

Demonstration.

Suppose the $\triangle BCD$ to be an Isosceles $\triangle$; That is, let $BC = CD$. Bisect the $\angle C$, or (which is all one) make CA Perpendicular to BD ; then will the $\angle$ on each Side of it, viz. $\angle BAC$ and $\angle DAC$ be Right Angles.



Therefore $\left\{ \begin{array}{l} \frac{1}{2} \angle C + \angle B = 90^\circ. \\ \frac{1}{2} \angle C + \angle D = 90^\circ. \end{array} \right\}$ Per Corol. to Theorem 4.

Consequently $\frac{1}{2} \angle C + \angle B = \frac{1}{2} \angle C + \angle D$ Per Axiom 5.
Subtract $\frac{1}{2} \angle C$ from both Sides of the Equation, and it will leave $\angle B = \angle D$. Per Axiom 2. Q. E. D.

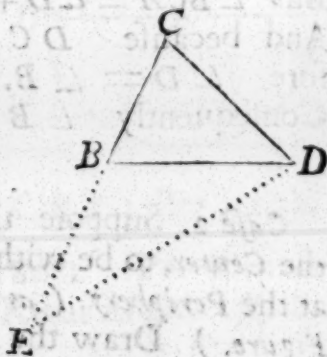
Corollary.

From hence it follows, that the Three Angles of an Equilateral Triangle, are Equal one to another.

THEOREM VII.

In every Plain Triangle, the Longest Side subtends the greatest Angle. (18. e. 1.)
Consequently, the greatest Angle of any Plain Triangle, is subtended by the Longest Side. (19. e. 1.)

This Theorem is evident by Inspection only; For let one of the Sides of any Plain Triangle, as CB , be produced, suppose to E ; joyn DE with a Right Line; Then 'tis evident, that because CE is now made Longer than the Side BC , therefore the $\angle$ at D is become Larger than it was before by the $\angle BDE$; and its plain, the longer the Side CE had been made, the $\angle$ at D would have been the more Inlarged.



THEOREM VIII.

If the Sides of Two Triangles are Equal, the Angles Opposite to those equal Sides will be Equal. (8. e. 1.)

The Truth of this Theorem is evident by the Two included Triangles in the 6th Theorem, for they have their respective Sides Equal, viz. $BC = CD$, $BA = DA$, and CA common to both $\triangle\triangle$. And it is there proved, That the $\angle$ opposite to those Equal Sides, are Equal, &c. which needs no further Proof.

Note, The Converse of this Theorem holds not true; for the Angles of Two Triangles may be Equal, and their Opposite or subtending Sides Unequal, as will appear at Theorem 12.

Corollary.

Hence it follows, That $\triangle\triangle$ mutually Equilateral, are also mutually Equiangular. And

That $\triangle\triangle$ mutually Equilateral, are Equal one to another. (4, and 26. e. 1.)

THEOREM IX.

An Angle at the Center of any Circle, is always Double to the Angle at the Periphery, when both the Angles stand upon the same Arch. (20. e. 3.) This Theorem hath three Varieties or Cases.

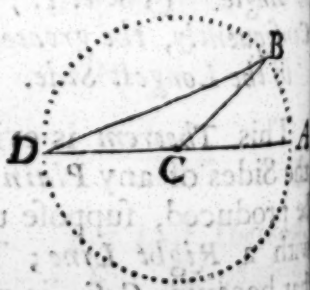
Demonstration.

Case 1. Let the Diameter DA , and the Line DB , be the Two Lines which form the $\angle$ at the Periphery; Draw the Radius BC , then is $\angle BCA$ the $\angle$ at the Center.

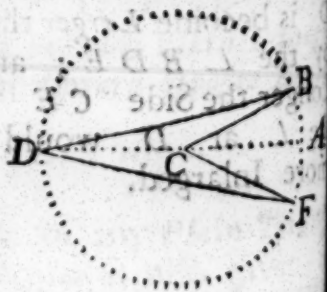
But $\angle BCA = \angle D + \angle B$. Per Theor. 5.

And because $DC = BC$, therefore $\angle D = \angle B$. Per Theorem 6.

Consequently $\angle BCA = 2 \angle D$.



Case 2. Suppose the $\angle BCF$ at the Center, to be within the $\angle BDF$ at the Periphery (as in the Annexed Figure.) Draw the Diameter DA , Then the $\angle BCA = 2 \angle BDA$, And the $\angle FCA = 2 \angle FDA$ } per Case 1
Add these Two Equations together,



Then will $\angle BCA + \angle FCA = 2\angle BDA + 2\angle FDA$ per Axiom 1.
 But $\angle BCA + \angle FCA = \angle BCF$.
 And $2\angle BDA + 2\angle FDA = 2\angle BDF$.
 Consequently $\angle BCF = 2\angle BDF$.

Case 3. Again, suppose the $\angle BCF$ at the Center, to be out of the $\angle BDF$ at the Periphery. From the Angular Point D at the Periphery, draw the Diameter DA .



Then $\angle FCA = 2\angle FDA$
 And $\angle BCA = 2\angle BDA$ } per Case 1.

Subtract this Last Equation from the other, and it will Leave

$\angle FCA - \angle BCA = 2\angle FDA - 2\angle BDA$ Per Axiom 2.

But $\angle FCA - \angle BCA = \angle FCB$. And $2\angle FDA - 2\angle BDA = 2\angle FDB$. Consequently $\angle FCB = 2\angle FDB$. Q. E. D.

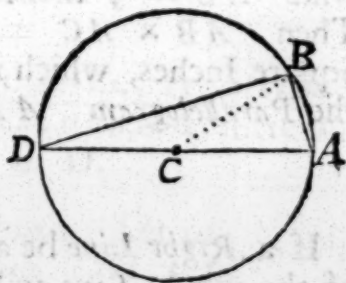
Corollary.

Hence its Evident, That all Angles at the Periphery, which stand on the same Segment or Arch of a Circle, or upon equal Arches, are Equal one to another. (21. e. 3.)

THEOREM X.

An Angle in a Semi-circle, is a Right Angle. (31. e. 3.)
 That is, if the Diameter of any Circle be the Side of a Triangle, and the Angle Opposite to that Side, be any where in the Circle's Periphery, it will be a Right Angle.

Demonstration.



Let DA be the Diameter, and DBA the $\triangle$, then $\angle B = 90^\circ$. Draw the Radius BC , then is the $\angle DBA = \angle D + \angle A$.

For $\angle CBD = \angle D$ and $\angle CBA = \angle A$

Per Theorem 6.

Therefore $\angle DBA = \angle CBD + \angle CBA$. Per Axiom 5.

Again $\angle DBA + \angle D + \angle A = 180^\circ$. Per Theorem 4.

Consequently $\angle DBA = 90^\circ$, or a Right Angle. Q. E. D.

Corollaries.

1. Hence it will be Easie to Conceive, that an *Angle* made in any *Segment* Less than a *Semi-circle*, will be *Obtuse* or Greater than a *Right Angle*.

2. And an *Angle* made in any *Segment* Greater than a *Semi-circle*, must consequently be *Acute*.

THEOREM XI.

In any *Right Angled Triangle*, the *Square* which is made of the *Hypotenuse*, or *Side subtending the Right Angle*, is *Equal* to both the *Squares* which are made of the *Sides* including the *Right Angle*. (47. e. 1.)

There are several ways of Demonstrating this Noble and Useful Theorem; but I presume none more Easie to be understood by a Learner, than that which I shall here propose; And in order thereto, it will be requisite to premise the following *Lemma's*.

Lemma 1.

A *Right Line* is said to be *Multiplied* with a *Right Line*, when either a *Square*, or other *Right-angled Parallelogram* is made of the *Two Lines*.

That is, the *Area* of any *Right-angled Parallelogram*, is equal to the *Product* of those *Numbers* which express the *Measure* of its *Sides*.

Thus, if $AB = 6$ Inches, &c.

And $AC = 3$ Inches;

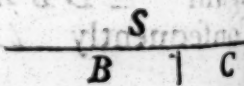
Then $AB \times AC = 6 \times 3 = 18$ Square Inches, which is the *Area* of the *Parallelogram* $ABCD$.



Lemma 2.

If a *Right Line* be any-wise Cut into *Two Parts*, the *Square* of the whole *Line* will be *Equal* to the *Squares* of each *Part*, and a double *Rectangle* or *Parallelogram* made of both the *Parts*. (4. e. 2.)

That is, if the *Line* S , be cut into the *Two Parts* B and C Then is $S = B + C$ but if both the *Sides* of the *Æquation* be *Involved*, it will be $SS = BB + 2BC + CC$



Lemma 3.

The Area of every Right-angled Triangle, is Half the Parallelogram made of its Base and Perpendicular.

For BC = the Area of the whole Parallelogram, by the First Lemma. And $\triangle BCH + \triangle b c H$ = the Parallelogram; But $B = b$, and $C = c$.

Therefore $\frac{1}{2}BC$ = the Area of each $\triangle$ viz. $\frac{1}{2}BC + \frac{1}{2}bc = BC$

These things being premised, let us suppose, the Triangle BCH to be a Right-angled Triangle. Viz. the Side C perpendicular to the Side B . Then will $BB + CC = HH$.

Demonstration.

Make a Square whose Side is $= B + C$, and draw the included Square whose Side is $= H$, as in the Scheme. Then will the Area of the great Square, be equal to the Area of the Four Triangles $+ HH$, but the Area of each $\triangle = \frac{1}{2}BC$. Per Lemma 3.

Therefore the $4 \triangle's = \frac{1}{2}BC \times 4 = 2BC$. Consequently, the Area of the great Square is $HH + 2BC$. Involve $B + C$, and it will be $BB + 2BC + CC =$ the Area of the great Square. Per Lemma 2.

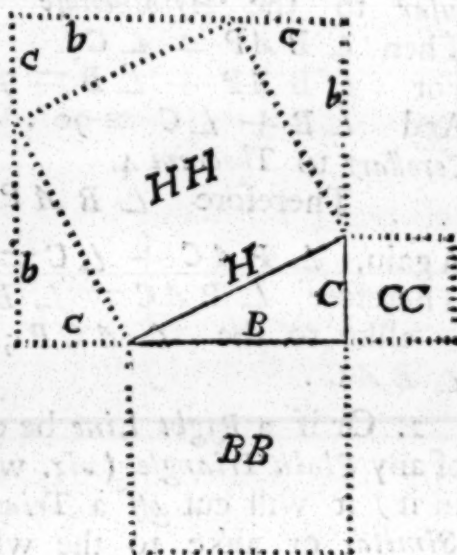
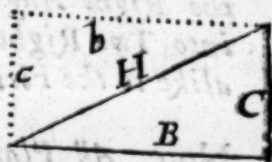
Consequently $HH + 2BC = BB + 2BC + CC$. Per Axiom 5. Subtract $2BC$ from both Sides of the Equation, and there will Remain $HH = BB + CC$. Q. E. D.

To Illustrate this Theorem by Numbers; Let us Suppose $C = 3$. $B = 4$. and $H = 5$. Then will $CC = 9$. $BB = 16$. and $HH = 25$. Consequently, $BB + CC = HH = 16 + 9 = 25$

Corollary.

From this admirable Theorem (said to be First Invented by Pythagoras) is deduced the Method of Adding, and Subtracting Squares, Parallelograms, Circles, &c.

THE-



THEOREM XII.

In any Right-angled Triangle, a Perpendicular being let fall from the Right Angle upon the Hypotenuse, will divide the Triangle into Two Right-angled Triangles, which will be both Similar or alike to the First Triangle, and to each other. (8. e. 6.)

Note, All Plain Triangles are said to be Similar, viz. alike, when each Single Angle of one of the Triangles, is Equal to each Single Angle of the other; but if any Two Single Angles of one Triangle, are Equal to Two Single Angles of the other; The Third Angles will be Equal. Per Theorem 4.

1. In the Right-angled $\triangle BAC$,
Let AP be supposed Perpendicular to the Hypotenuse BC ,
Then $\angle BAP = \angle C$.

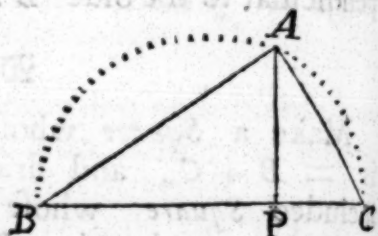
For $\angle BAP + \angle B = 90^\circ$:

And $\angle B + \angle C = 90^\circ$. Per

Corollary to Theorem 4.

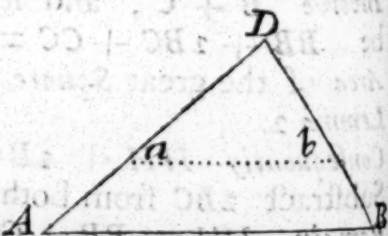
Therefore $\angle BAP = \angle C$. Per Axiom 5.

Again, $\angle PAC + \angle C = 90^\circ$. and $\angle B + \angle C = 90^\circ$.
Therefore $\angle PAC = \angle B$, &c. Consequently the $\triangle BAP$, is alike to the $\triangle ACP$; And each is alike to the whole $\triangle BAC$.



2. Or if a Right Line be drawn Parallel to one of the Sides of any Plain Triangle (viz. within it) it will cut off a Triangle Similar or alike to the whole Triangle. Thus,

In the $\triangle ABD$, draw the Right Line ab Parallel to the Side AB ; Then will the Included $\triangle adb$, be like the $\triangle ADB$. For $\angle a = \angle A$, and $\angle b = \angle B$. Per Theorem 3. and $\angle D$ is common to both the Triangles: Ergo, &c.



THEOREM XIII.

If Two Triangles are alike, their like Sides will be Proportional.

That is, those Sides which subtend the equal Angles, as also those Sides which are about the equal Angles, will be Proportional to each other; And consequently, if any Two Triangles have their Sides Proportional, their Angles are equal. (4, 5, 6, 7. e. 6.)

Demon

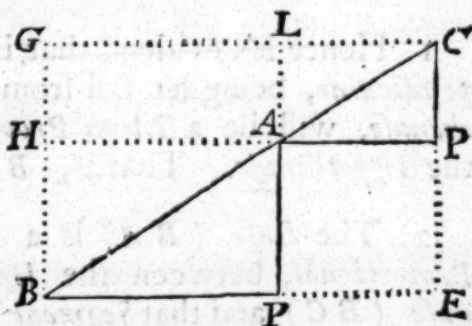
Demonstration.

Let the *Similar Triangles* in the Scheme of the Last Theorem, be here proposed again.

Then it will be $BP : AP :: AP : CP$. According to this Theorem. Ergo $BP \times CP = AP \times AP$.

Demonstration.

Let us suppose the aforesaid *Right-Angled* $\triangle BAC$, cut through the *Perpendicular* AP , and there opened until the *Sides* BA and CA become one *Right Line*: Let the *Sides* BP and CP be continued until they meet in E ; then compleat the *Parallelograms* by drawing the *Parallel Lines* GLC , HAP , GHP , and LAP , as in the Figure:



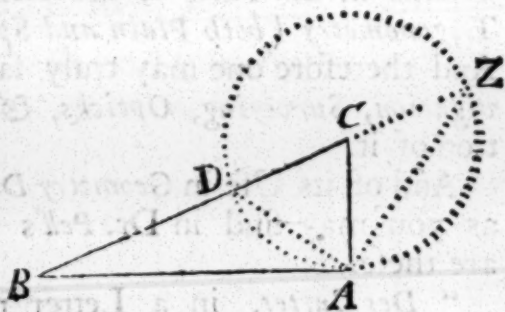
Then it's evident, that the $\triangle BHA = \triangle BPA$, and the $\triangle CPA = \triangle CLA$; also that the $\triangle BEC = \triangle BGC$, because all their respective *Sides* are *Equal*:

But the $\triangle BHA + \triangle CLA + \square HGLA = \triangle BPA + \triangle CPA + \square APEP$. Now if from both *Sides* of this Equation there be Subtracted the equal $\triangle$, there will Remain $\square HGLA = \square APEP$.

But $\square HGLA = BP \times CP$, and $\square APEP = AP \times AP$. Consequently $BP : AP :: AP : CP$. Which was to be proved.

Or otherwise, Thus.

Suppose the $\triangle BAC$ to be *Right-angled* at A , upon the Point C with the *Radius* CA , describe a *Circle*, and continue the *Hypotenuse* BC to D ; joyn ZA and AD with *Right Lines*; then will the $\triangle BAD$ be like to the $\triangle ZCA$.



$\angle DAB + \angle DAC = 90^\circ$: By Construction.
 $\angle ZAC + \angle DAC = 90^\circ$. Per Theorem 10.
 Therefore $\angle DAB + \angle DAC = \angle ZAC + \angle DAC$. Per Axiom 5.
 Subtract $\angle DAC$ from both *Sides* of the Equation, and there will Remain $\angle DAB = \angle ZAC$. But $\angle ZAC = \angle CZA$
 R r Per

Per Theorem 6. And $\angle B$ is common to both $\triangle \triangle$
Therefore $\angle B D A = \angle B A Z$. Per Theorem 4.
Consequently $\triangle B A D$, is like to $\triangle B Z A$.

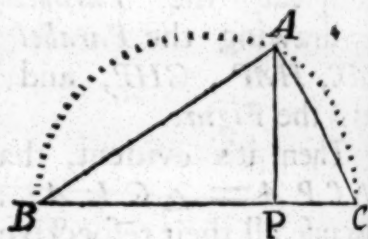
Let the Sides $\begin{cases} BA = b \\ BC = h \\ CA = c \end{cases} \left\{ \begin{array}{l} \text{Then } bb + cc = hh. \text{ Per Theor. II.} \\ \text{Consequently } bb = hh - cc \\ \text{Which gives the following Analogy,} \end{array} \right.$
Viz. $b : h + c :: h - c : b$. That is, $BA : BZ :: BD : BA$.
Q. E. D.

Corollaries.

1. Hence it's evident, that in any *Right-angled Triangle*, a *Perpendicular*, being let fall from the *Right Angle* upon the *Hypothenufe*, will be a *Mean Proportional* between the *Segments* of the *Hypothenufe*. That is, $BP : PA :: PA : PC$.

2. The *Base* (BA) is a *Mean Proportional*, between the *Hypothenufe* (BC) and that *Segment* of the *Hypothenufe* next to the *Base* (viz. BP .) That is, $BC : BA :: BA : BP$.

3. The *Cathetus* (AC) is a *Mean Proportional* between the *Hypothenufe* (BC) and that *Segment* of the *Hypothenufe* next to the *Cathetus* (viz. PC .) That is, $BC : AC :: AC : PC$.



Scholium.

I have been more large upon this most excellent Theorem, in giving a double *Demonstration* of it, because it is so universally Useful in all Parts of the *Mathematicks*: For the *Business* of *Trigonometry* (both *Plain* and *Spherical*) wholly depends upon it. And therefore one may truly say, that *Astronomy*, *Dialing*, *Navigation*, *Surveying*, *Opticks*, &c. depend upon a due Application of it.

And of its Use in *Geometry* *Des Cartes* takes particular Notice as you may find in *Dr. Pell's Algebra* Page 65. whose Words are these.

“ *Des Cartes*, in a Letter not yet Printed, writes thus;
“ searching the Solution of Geometrical Questions, I always
“ make use of Lines Parallel, and Perpendicular, as much as
“ possible, (he means, as many Lines as are Useful) and I con-
“ sider no other Theorems but these Two [the Sides of Right
“ Triangles have like Proportion] And [In Rectangle Triangles]

"the Square of the greatest Side, is Equal to the Squares of the Two other Sides]. And I am not afraid to suppose many unknown Quantities, that I may reduce the proposed Question to such Terms, as that it depends on no other Theorems but these Two."

This I thought convenient to insert, that the young Learner may see how the Great Des Cartes Esteem'd of these Two Theorems, viz. the Last, and Theorem 11. for intruth, all the precedent Theorems are only (as it were) Preparatives to these Two.

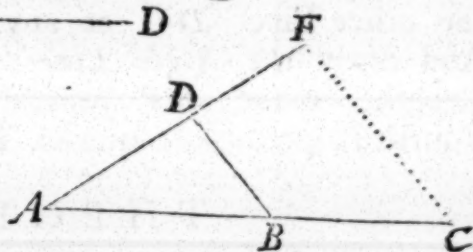
This Last Theorem Demonstrates the Reason of the Method used in finding out Proportional Lines; as in the Three following Problems.

PROBLEM I.

Two Right Lines being given; To find a third Line in Proportion to them (11. e. 6.)

Let the Two Lines be $\begin{cases} A \text{ --- } B \\ A \text{ --- } D \end{cases}$

Set the Two given Lines at any Angle in the Point A, and produce the Line AD to C, making $BC = AD$; Joyn the Points BD with a Right Line, and draw CF Parallel to



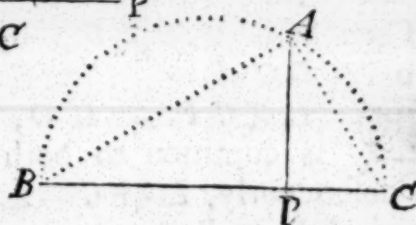
BD; Then will the $\triangle ABD$ be like the $\triangle ACF$. Therefore $AB : BC (= AD) :: AD : DF$, which is the Third Proportional required.

PROBLEM II.

Two Right Lines being given; To find a Mean Proportional Line between them. (13. e. 6.)

Let the given Lines be $\begin{cases} B \text{ --- } P \\ P \text{ --- } C \end{cases}$

Joyn the Two given Lines into one, viz. make $BC = BP + PC$, and upon BC, as Diameter, describe a Semicircle; Then upon the Point P, where the Two Lines meet,



erect a Perpendicular to touch the Circle's Periphery, as PA, And

And it will be the *Mean Proportional* required.

$$\text{viz. } BP : AP :: AP : PC.$$

By this *Problem*, it is easie to Conceive how to make a *Square* Equal to any given *Parallelogram*. (14. e. 6.)

For if BP be the *Length*, and PC the *Breadth* of the given *Parallelogram*; Then will AP be the Side of the *Square*, Equal in *Area* to that *Parallelogram*.

PROBLEM III.

Three Right Lines being given; To find a Fourth Proportional Line (12. e. 6.)

Suppose the Three Lines $\begin{cases} A \text{---} B \\ A \text{---} D \\ D \text{---} C \end{cases}$

upon the Longest Line AB .

Set off the next Longest Line

AD , viz. make $DB = AB - AD$.

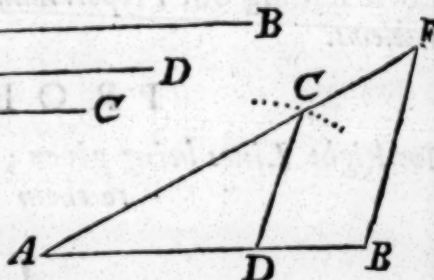
Then upon the Point D , set

the other Line DC at any Angle, either Right or Oblique;

And draw the Right Line AC , continuing it a sufficient

Length; make BF parallel to DC ; and it will be the

Fourth Proportional required. That is, $AD : DC :: AB : BF$.



THEOREM XIV.

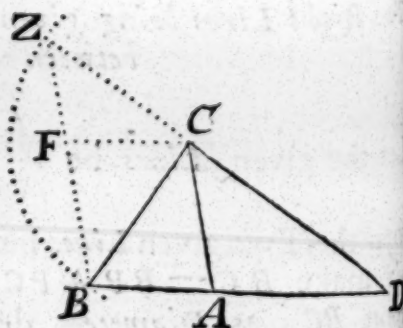
If any Angle of a Plain Triangle be Bifected (viz. divided into Two equal Angles) with a Right Line (viz. As CA is supposed to do the Angle BCD) it will cut the Opposite Side (viz. BD) in Proportion to the other Two Sides of the Triangle. (3. e. 6.)

Demonstration.

Produce the Side DC , until $CZ = CB$; Joyn the Points ZB with a Right Line, and draw the Line FC parallel to BD . Then will $\triangle CZF$, be like to $\triangle DCA$.

For $\angle ZCF = \angle D$, and $\angle Z$ is common to both $\triangle$'s. Consequently, $\angle ZFC = \angle CAD$ and $FC = BA$.

Therefore $BA (= FC) : BC (= CZ) :: AD : CD$. Q.E.D.



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THEOREM XV.

If Two Right Lines (howsoever drawn) within a Circle, do cut each other; the Rectangle made of the Segments (or Parts) of the one Line, will be Equal to the Rectangle made of the Segments (or Parts) of the other Line. (35. e. 3.)

That is, If Two Lines as AB , and CD , do cut each other in any Point, as at x . Then will $Ax \times Bx = Dx \times Cx$,

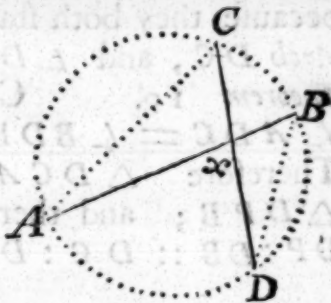
Demonstration.

Joyn the Points AC , and BD with Right Lines. Then will the $\triangle CxA$ be like to the $\triangle BxD$. For $\angle B = \angle C$, and $\angle A = \angle D$. Per Corollary to Theorem 9.

And $\angle Ax C = \angle Bx D$. Per Theor. 2.

Therefore it will be $Ax : Dx :: Cx : Bx$. Per Theorem 13.

Consequently, $Ax \times Bx = Dx \times Cx$. Q. E. D.



THEOREM XVI.

If Two Right Lines, are so drawn within a Circle, as being continued, they will meet in a Point out of the Circle's Periphery; the Rectangle made of one whole Line, and its Part out of the Circle, will be Equal to the Rectangle of the other whole Line, and its Part out of the Circle. (36, 37. e. 3.)

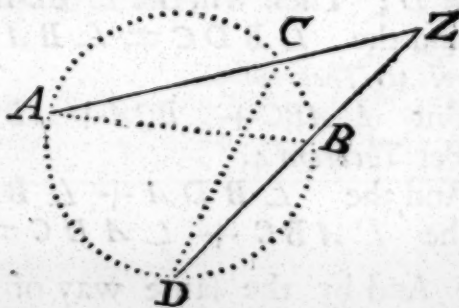
That is, if the Lines AC and DB , be continued unto the Point Z .

Then will $AZ \times CZ = DZ \times BZ$.

Demonstration.

Draw the Lines AB and CD , then will $\triangle CZD$ be like to the $\triangle BZA$, for $\angle A = \angle D$, and $\angle Z$ is common to both $\triangle\triangle$. Consequently $\angle ABZ = \angle DCZ$. Per Theorem 4.

Therefore $AZ : BZ :: DZ : CZ$. Ergo, $AZ \times CZ = DZ \times BZ$.



THEOREM XVII.

If from any Angle of a Plain Triangle Inscribed in a Circle : there be let fall a Perpendicular upon the Opposite Side (as DP)

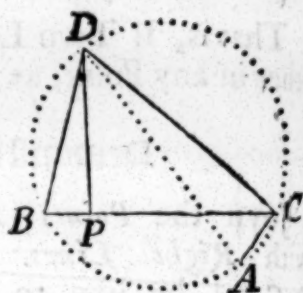
As

As that Perpendicular, Is in Proportion to one of the Sides including the Angle : So is the other Side Including the Angle : To the Diameter of the Circle.

Demonstration.

Let BCD be the proposed $\triangle$. From the $\angle$ at D , draw the Diameter DA ; then will $\angle A = \angle B$, because they both stand upon the same Arch DC , and $\angle DCA = 90^\circ$. Per Theorem 10. Consequently the $\angle ADC = \angle BDP$. Per Theorem 4. Therefore $\triangle DCA$ is like to the $\triangle DPB$; and therefore

$DP : DB :: DC : DA$, Or, $DP : DC :: DB : DA$. Q. E. D.



THEOREM XVIII.

If any Quadrangle (that is, a Trapezia) be Inscribed within a Circle; the Two opposite Angles taken together, are Equal to Two Right Angles, viz. 180° . (22. e. 3.)

That is, in the Quadrangle $ABCD$, the $\angle A + \angle C = 180^\circ$. And the $\angle B + \angle D = 180^\circ$.

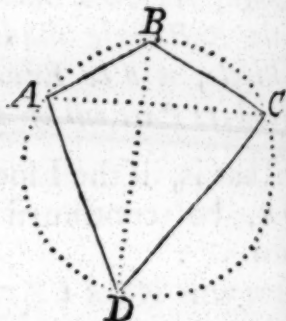
Demonstration.

Draw the Two Diagonals AC , and BD ; Then will the $\angle BDA = \angle BCA$, and the $\angle BDC = \angle BAC$. Per Corol. to Theorem 9.

But $\angle ABC + \angle BCA + \angle BAC = 180^\circ$. Per Theorem 4.

And the $\angle BDA + \angle BDC = \angle ADC$. Therefore the $\angle ABC + \angle ADC = 180^\circ$.

And by the same way of Arguing, it may be proved, that the $\angle BAD + \angle BCD = 180^\circ$. Q. E. D.



THEOREM XIX.

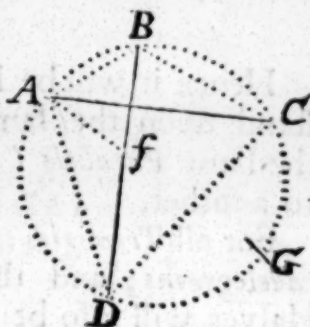
If in any Quadrangle Inscribed within a Circle, there be drawn Two Diagonals, As AC and BD , the Rectangle made of the Two Diagonals, will be Equal to both the Rectangles made of the opposite Sides of the Quadrangle.

That is, $AC \times BD = AB \times CD + AD \times BC$.

Demonstration.

Demonstration.

Make the Arch $DG = \text{Arch } BC$, and from the Points A, G , draw the Line Af , and it will Form the $\triangle AfD$, like to the $\triangle ABC$. For the $\angle fAD = \angle BAC$ because the Arches DG , and BC are Equal.



Again, the $\angle fDA = \angle BCA$, because they both stand upon the Arch AB . Consequently, the $\angle AfD = \angle ABC$. Per Theorem 4.

Therefore it will be $AC : BC :: AD : Df$. Per Theor. 13.

$$\text{Ergo } \left\{ \frac{BC \times AD}{AC} = Df. \right.$$

Again, the $\triangle Baf$, and $\triangle ACD$, are alike. For $\angle ABf = \angle ACD$ and $\angle Baf = \angle CAD$, because the $\angle fAD = \angle BAC$. And the $\angle Caf$ is common to both $\triangle$'s. Consequently, the $\angle AfB = \angle ADC$.

Therefore $AC : CD :: AB : Bf$. Per Theorem 13.

$$\text{Ergo } \left\{ \frac{CD \times AB}{AC} = Bf. \text{ But } Df + Bf = BD. \right.$$

Consequently, $BC \times AD + CD \times AB = BD \times AC$.

Q. E. D.

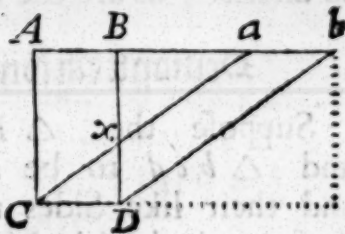
THEOREM XX.

All Parallelograms (whether Right or Oblique-angled) that stand upon the same Base, or upon Equal Bases, and betwixt the same Parallels, are Equal one to another. (35, and 36. e. 1.)

That is, $\square ABCD = \square abCD$.

Demonstration.

Because $AB = CD = ab$, by Supposition. Therefore $Aa = Bb$, for Ba is common to both. And because $AC = BD$, and the $\angle A = \angle B$, therefore the $\triangle ACa = \triangle BDb$; and if from both $\triangle$'s there be taken the $\triangle Bxa$ common to both; there will Remain the Trapeziums $ABxC = abxD$. Per Axiom 5.



But

But Trapezium $AB \times C + \triangle C \times D = \square ABCD$.

And Trapezium $ab \times D + \triangle C \times D = \square abCD$.

Consequently, $\square ABCD = \square abCD$. Q. E. D.

Corollary.

Hence it will be Easie to Conceive, that all *Triangles* which stand upon the same *Base*, or upon *Equal Bases*, and between the same *Parallels* (viz. *having the same Height*) are *Equal* one to another. (37, and 38. e. 1.)

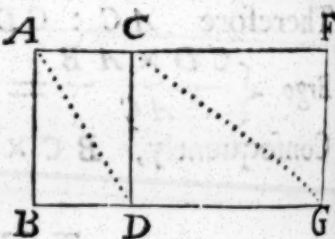
For all *Triangles* are the *Halves* of their *Circumscribing Parallelograms*; and therefore, if the *Wholes* be *Equal*, their *Halves* will also be *Equal*.

THEOREM XXI.

Parallelograms (and consequently Triangles) which have the same Height, have the same Proportion one to another, as their Bases have. (1. e. 6.)

Demonstration.

Draw AF Parallel to BG , and draw AB , CD , FG Perpendiculars to them.



Then will $BD \times AB = \square ABCD$.

And because $CD = AB$, there-

fore $DG \times AB = \square CDFG$:

But $BD : DG :: BD \times AB : DG \times AB$.

And consequently $\triangle ABD : \triangle CDG :: BD : DG$, &c.

Q. E. D.

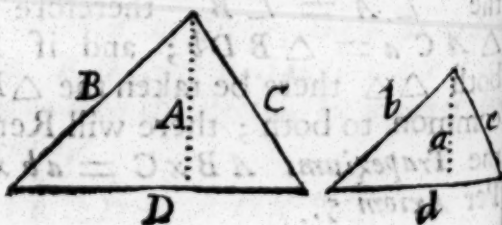
THEOREM XXII.

Like Triangles are in Duplicate Ratio to that of their Homologous Sides. (19. e. 6.)

That is, the *Area's* of like *Triangles*, are in *Proportion* one to another, as are the *Squares* of their like *Sides*.

Demonstration.

Suppose the $\triangle BCD$ and $\triangle bcd$ to be alike, and their like Sides to be those marked with the same Letters.



Let A and a , be Perpendiculars to the Two Bases D and d .
 Then $\frac{1}{2}DA =$ the Area of $\triangle BCD$ } Per Lemma 3. Page 303.
 And $\frac{1}{2}da =$ the Area of $\triangle bcd$ }

But	1	$B : b :: D : d$	} &c. Per Theorem 13.
And	2	$B : b :: A : a$	
Conseq.	3	$D : d :: A : a$	
3 ::	4	$Da = dA$	
4 $\times \frac{1}{2}Dd$	5	$\frac{1}{2}DDda = \frac{1}{2}DddA$	Per Axiom 3.
5, hence	6	$DD : dd :: \frac{1}{2}DA : \frac{1}{2}da$	And so for other Sides

Q. E. D.

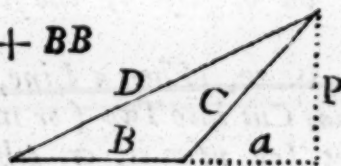
THEOREM XXIII.

In every Obtuse-angled Triangle, (as BCD) the Square of the Side subtending the Obtuse Angle (as D) is greater than the Squares of the other Two Sides (B and C) by a double Rectangle made of one of the Sides (as B) and the Segment or Part of that Side produced (as a) until it meet with the Perpendicular (P) Let fall upon it. (12. e. 2.)

That is, $DD = BB + CC + 2Ba$.

Demonstration.

First	1	$DD = PP + aa + 2Ba + BB$
And	2	$CC = PP + aa$
- 2	3	$DD - CC = 2Ba + BB$
+ CC	4	$DD = BB + CC + 2Ba$



Q. E. D.

Corollary.

Hence it's evident, that if the Sides of any Obtuse-angled Triangle are given; the Segment (a) of the Side produced, or the Perpendicular (P) may be easily found.

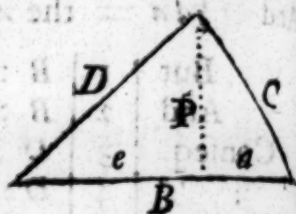
THEOREM XXIV.

A Perpendicular (as P) be let fall in any Acute-angled Triangle (as BCD) the Square of either of the Two Sides (as D) is Less than the Squares of the other Side, and that Side upon which the Perpendicular falls (viz. C and B) by a double Rectangle made of the Side B , and that Segment or Part of it (viz. a) which lies next to the Side C . (13. e. 2.)

That is, $DD + 2Ba = BB + CC$.

Demonstration.

First	1	$DD = PP + ee$	} Per Theo. 11.
And	2	$CC = PP + aa$	
But	3	$B - a = e$	Per Figure.
3 $\odot$ 2	4	$BB - 2Ba + aa = ee$	
4 $- aa$	5	$BB - 2Ba = ee - aa$	
1 $- 2$	6	$DD - CC = ee - aa$	
5, 6	7	$DD - CC = BB - 2Ba$	
7 $+$	8	$DD + 2Ba = BB + CC$	



Q. E. D.

Corollary.

Hence it follows, that if the *Sides* of any *Acute-angled Triangle* be known, the *Perpendicular* (*P*) and the *Segments* of the *Side* whereon it falls (*viz.* *a*, *e*) may be easily found.

C H A P. IV.

The Solution of several Easie Problems in Plain Geometry; whereby the Learner may (in part) perceive the Application or Use of the foregoing Theorems.

Note, When a Line, Or the Side of any Plain Triangle, is any way Cut into Two (or more) Parts, either by a Perpendicular Line let fall upon it, or otherwise; those Parts are usually called Segments; and so much as one of those Parts is Longer than the other, is called the Difference of the Segments.

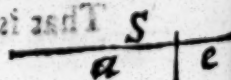
And when any Side of a Triangle, or any Segment of its Side is given, it is usually marked with a small Line cross it, thus $\perp$ and those Sides, or Parts of Sides that are sought, are marked with four Points, thus $\dots$

Problem 1.

To Cut or Divide a given Right Line (as *S*) into Extreme and Mean Proportion. (11. e. 2.)

That is, to Divide a Line so, that the Square of the Greater Segment (or Part) *a*, may be Equal to the Rectangle made of the whole Line *S*, and the Lesser Segment *e*.

Viz.	1	$Se = aa$	by the Problem.
And	2	$S - a = e$, for $S = a + e$	

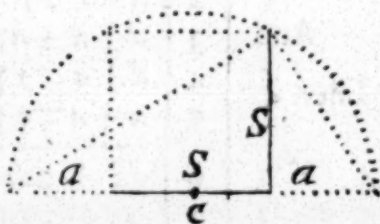


1 ÷

1	$\div S$	3	$\frac{aa}{S} = e$
2 and 3		4	$\frac{aa}{S} = S - a$. Per Axiom 5.
4	$\times S$	5	$aa = SS - Sa$
5	$+ Sa$	6	$aa + Sa = SS$
6, Solved		7	$a = \sqrt{SS + \frac{1}{4}SS} : -\frac{1}{2}S$. Vide Pages 195, 196.

Note, The last Problem cannot be truly Answer'd by Numbers; but Geometrically it may be perform'd thus.

1. Make a Square whose Side is $= S$ the given Line, and Bisect one of its Sides in the Middle, as at C; upon the Point C describe such a Semi-circle, as will pass through the Remotest Points of the Square, and Compleat its Diameter.



2. Then will either Part of the Diameter, on each End of the Side S , be $= a$, the Greater Segment sought.

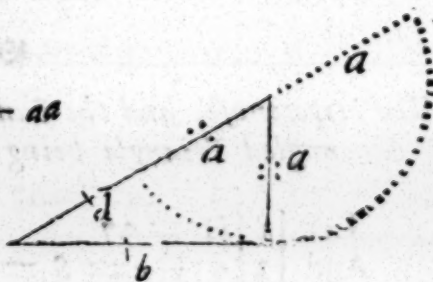
For $a + S : S :: S : a$. Per Theorem 13.

Ergo, $aa + Sa = SS$. Which was to be done.

Problem 2.

The Base of any Right-angled Triangle, and the Difference between the Hypotenuse and Cathetus being given; To find the Cathetus, &c.

Let	1	$b = 72$
	2	$d = 32$
And	3	$a = \text{Cathetus sought}$
Then	4	$bb + aa = dd + 2da + aa$
		Per Theor. 11.
$- aa$	5	$bb = dd + 2da$
$- dd$	6	$2da = bb - dd$
$\div 2d$	7	$a = \frac{bb - dd}{2d} = 65$



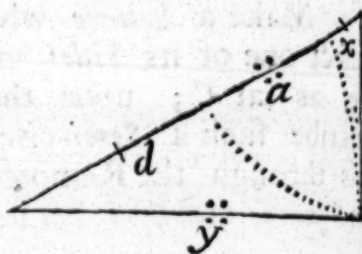
Or 8 $b : d + 2a :: d : b$ Per Theorem 13.
 $\therefore$ 9 $bb = dd + 2da$. As before at the 5th Step.

Here you see, that either way *Raises* the same *Æquation*; Neither is there any constant *Method* or *Road* to be observ'd, in *Solving Geometrical Problems*; but every one makes use of such *Ways* and *Theorems* as happen to come first into their Mind, the *Result* being every way the same.

Problem 3.

The Difference between the Base and Hypotenuse of any Right-angled Triangle, and the Difference between the Cathetus and Hypotenuse being both given; To find the Triangle.

Let	{	1	$d = 32$	
		2	$x = 25$	
And		3	$d + x + a = \text{the Hypot.}$	
Then	{	4	$d + a = y$	} by the Prob.
		5	$x + a = e$	
4	⊙	2	6	$dd + 2da + aa = yy$
5	⊙	2	7	$xx + 2xa + aa = ee$
3	⊙	2	8	$dd + 2dx + 2da + 2xa + xx + aa = \square \text{ Hypotenuse}$
6	+	7	9	$dd + 2da + 2xa + xx + 2aa = yy + ee$



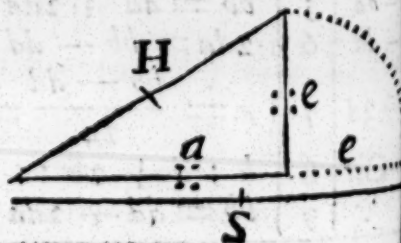
The Two Last Steps are Equal; per Theorem 11. Consequently, if those Things that are Equal in both, be taken away, the Remainders will be Equal. Per Axiom 2.

That is,	10	$aa = 2dx = 1600$	
10 <i>uw</i> 2	11	$a = \sqrt{2dx} = 40$	
1 + 11	12	$d + a = 72 = y$	The <i>Base</i> .
2 + 11	13	$x + a = 65 = e$	The <i>Cathetus</i> .
1 + 2 + 11	14	$d + x + a = 97$	The <i>Hypothenufe</i> .

Problem 4.

The Hypotenuse, and the Sum of the other Two Sides, of any Right-angled Triangle being given; Thence to find the Sides.

Let	1	$H = 97$	
And	2	$a + e = S = 137$	
per Fig.	3	$aa + ee = HH$	
2 ⊙ 2	4	$aa + 2ae + ee = SS$	
4 — 3	5	$2ae = SS - HH$	
3 — 5	6	$aa - 2ae + ee = 2HH - SS$	
6 uw 2	7	$a - e = \sqrt{2HH - SS}$	

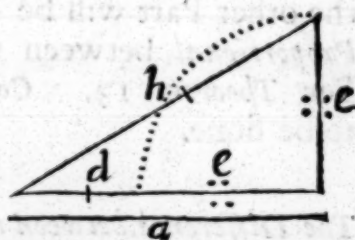


$2 + 7$	8	$2a = S + \sqrt{2HH - SS} = 144$
$8 \div 2$	9	$a = \frac{S + \sqrt{2HH - SS}}{2} = 72$ The Base required.
$2 - 9$	10	$e = \frac{S - \sqrt{2HH - SS}}{2} = 65$ The Cathetus.

Problem 5.

The Hypotenuse, and the Difference of the other Two Sides, of any Right-angled Triangle being given; To find the Sides.

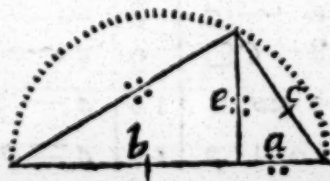
Let	1	$b = 97$ As before,
And	2	$a - e = d = 7$ Quere a .
per Fig.	3	$aa + ee = bb$
$2 \odot 2$	4	$aa - 2ae + ee = dd$
$3 - 4$	5	$2ae = bb - dd$
$3 + 5$	6	$aa + 2ae + ee = 2bb - dd$
$6 \div 2$	7	$a + e = \sqrt{2bb - dd}$
$2 + 7$	8	$2a = d + \sqrt{2bb - dd} = 144$
$8 \div 2$	9	$a = 72$
$7 - 2$	10	$2e = \sqrt{2bb - dd} - d = 130$
$1 \div 2$	11	$e = 65$



Problem 6.

In any Right-angled Triangle, either the Base, or Cathetus, and the Alternate Segment of the Hypotenuse (made by a Perpendicular let fall from the Right Angle) being given; To find the other Segment, &c.

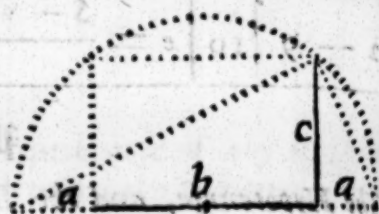
Let	1	$c = 45$ The Cathetus
And	2	$b = 48$ The Alternate Seg.
Then	3	$b : e :: e : a$ Quere a
$3 \therefore$	4	$ba = ee$
Again	5	$cc - aa = ee$ Per Theor. 11.
$4, 5$	6	$ba = cc - aa$
$6 + aa$	7	$aa + ba = cc$
$7, C \square$	8	$aa + ba + \frac{1}{4}bb = cc + \frac{1}{4}bb$
$8 \div 2$	9	$a + \frac{1}{2}b = \sqrt{cc + \frac{1}{4}bb}$
$9 - \frac{1}{2}b$	10	$a = \sqrt{cc + \frac{1}{4}bb} - \frac{1}{2}b = 27$ And so on for e , &c.



I shall

I shall now shew the Geometrical Construction, or Solution of the Three Cases of Quadratick Equations, promised in Page 202. Let the First Example be that above, viz. $aa + ba = co$, Case 1.

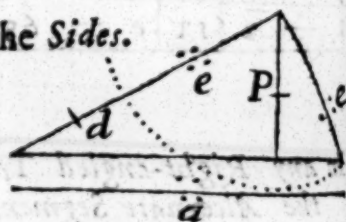
Make the Co-efficient b , and the Root of the Resolvend, (which is here) c , into a Right-angled Parallelogram (per Problem 13. Chap. 2.) And upon the middle Point of the Side $= b$, describe such a Semi-circle, as will pass through the Remotest Points (or Angles) of the Parallelogram, Compleating its Diameter, as in the Annexed Scheme. Then will either Part of the Diameter on each End b , be Equal to a , the other Part will be $a - b$, And the Side c will be a Mean Proportional between them. That is, $a + b : c :: c : a$. Per Theorem 13. Consequently $aa + ba = cc$. Which was to be done.



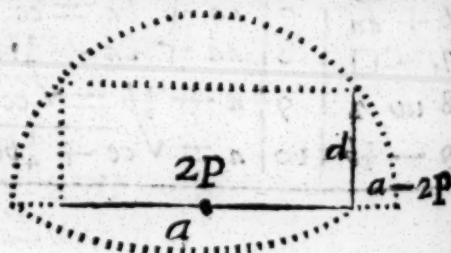
Problem 7.

The Difference between the Base and Cathetus of any Right-angled Triangle, and the Perpendicular let fall from the Right Angle upon the Hypotenuse, being given; Thence to find the Hypotenuse, &c.

Let	1	$d = 15$	The Diff. of the Sides.
And	2	$p = 36$	
Quere a	3	$a =$	The Hypotenuse.
per Fig.	4	$d + e : p :: a : e$	
4	5	$de + ee = pa$	
Again	6	$dd + 2de + 2ee = aa$	Per Theorem 11.
5 $\times$ 2	7	$2de + 2ee = 2pa$	
6 $-$ 7	8	$dd = aa - 2pa$	Case 2.
8 $C\Box$	9	$aa - 2pa + pp = dd + pp = 1521$	
9 uv 2	10	$a - p = \sqrt{dd + pp} = 39$	
10 $+ p$	11	$a = p + \sqrt{dd + pp} = 75$	&c. for e . per Step 5.



The Geometrical Construction of this Case 2. viz. $aa - 2pa = dd$ may be perform'd in the very same manner as the Last Case was. That is, by making a Right-angled Parallelogram of the Co-efficient $2p$, and the $\sqrt{dd}$, viz. d , &c. as in the Annexed Figure.



Then

Then will the Greater Part of the Diameter to one End of the Parallelogram, be $= a$; and the Lesser Part will be $a - 2p$.

For $4 : d :: d : a - 2p$. Per Theorem 13.

Consequently, $aa - 2pa = dd$. Which was to be done.

Problem 8.

The Hypotenuse of any Right-angled Triangle, and the Perpendicular let fall from the Right-angle upon the Hypotenuse, being given; To find the Greater Segment of the Hypotenuse, &c.

Let 1 $b = 73$ The Hypotenuse.
And 2 $p = 36$
Then 3 $a + e = b$ Quere a .

per Fig. 4 $a : p :: p : e$

4 $\therefore$ 5 $\frac{pp}{a} = e$

3 $- a$ 6 $b - a = e$

5, 6 7 $b - a = \frac{pp}{a}$

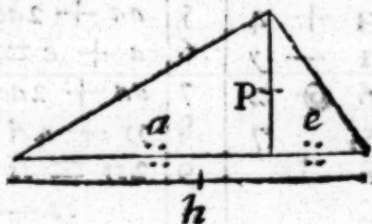
7 $\times a$ 8 $ba - aa = pp$ Case 3.

8 $\div$ 9 $aa - ba = -pp$

9 $\square$ 10 $aa - ba + \frac{1}{4}bb = \frac{1}{4}bb - pp = 110,25$

10 $uv 2$ 11 $a - \frac{1}{2}b = \sqrt{\frac{1}{4}bb - pp} = 10,5$

11 $+\frac{1}{2}b$ 12 $a = \frac{1}{2}b + \sqrt{\frac{1}{4}bb - pp} = 48$ Or $a = 27$



The Geometrical Construction of Case, 3. viz. $ba - aa = pp$,

may be thus perform'd. Draw a Right Line (of any convenient Length at Pleasure) and near its Middle Erect a Perpendicular $= p$, viz. of the same Length with the Root of the Resolvend; From the Top Point or upper End of that Perpendicular, set off Half the Length of the Co-efficient,

viz. $\frac{1}{2}b$, and upon the Point where $\frac{1}{2}b$ just touches the First Line, (with the same Distance) describe a Semi-circle; then will its Diameter b , be cut by the Perpendicular p into Two Segments, which are the Two Values of the Root a , viz. the Greater and Lesser Roots, both taken together being always Equal to the Co-efficient, (vide Page 201.)



For $b - a : p :: p : a$ Per Theorem 13.

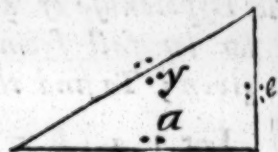
Ergo $ba - aa = pp$ Which was to be done.

problem

Problem 9.

The Perimeter (viz. the Sum of all the Three Sides) of any Right-angled Triangle, and its Area being given; Thence to find each Side.

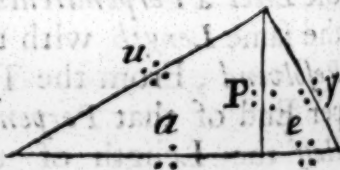
Viz. Let	1	$a + e + y = s = 234$	The Sum of the Sides.
And	2	$\frac{1}{2}ae = A$	The Area = 2340.
Again	3	$aa + ee = yy$	Per Figure.
2 x 4	4	$2ae = 4A$	
3 + 4	5	$aa + 2ae + ee = yy + 4A$	
1 - y	6	$a + e = s - y$	
6 ⊙ 2	7	$aa + 2ae + ee = ss - 2sy + yy$	
5, 7	8	$yy + 4A = ss - 2sy + yy$	
8 ±	9	$2sy = ss - 4A = 45396$	
9 ÷ 2s	10	$y = \frac{ss - 4A}{2s} = \frac{1}{2}s - \frac{2A}{s} = 97$	The Hypotenuse.
6, 10	11	$a + e = s - y = 137$	
3 - 4	12	$aa - 2ae + ee = yy - 4A = 49$	
12 √ 2	13	$a - e = \sqrt{49} = 7$	
11 + 13	14	$2a = 137 + 7 = 144$	
13 ÷ 2	15	$a = 72$	The Base.
11 - 15	16	$e = 137 - 72 = 65$	The Cathetus.



Problem 10.

In any Right-angled Triangle, a Perpendicular being let fall from the Right Angle, upon the Hypotenuse; If the Sum of each Segment, when Added to its Adjacent or next Side, be given; Thence to find each Side and the Segments.

Viz. If	1	$a + u = s = 108$	
And	2	$e + y = z = 72$	
To find		$a, e, u, y, \text{ and } p$	
1 - a	3	$u = s - a$	
3 ⊙ 2	4	$uu = ss - 2sa + aa$	
4 - aa	5	$uu - aa = ss - 2sa = pp$	
2 - e	6	$z - e = y$	
6 ⊙ 2	7	$zz - 2ze + ee = yy$	
7 - ee	8	$zz - 2ze = yy - ee = pp$	
5, 8	9	$zz - 2ze = ss - 2sa$	
per Fig.	10	$a : p :: p : e$	
10 ∴	11	$ae = pp$	
5, 11	12	$ae = ss - 2sa$	



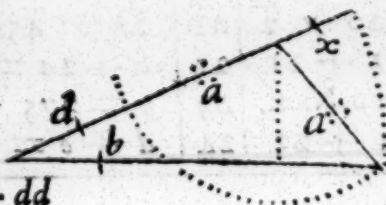
12 ÷ 6

12 ÷ a	13	$e = \frac{ss - 2sa}{a}$
13 × 27	14	$27e = \frac{27ss - 47sa}{a}$
9 + 14	15	$27 = ss - 2sa + \frac{27ss - 47sa}{a}$
15 × a	16	$27a = ssa - 2saa + 27ss - 47sa$
16 ÷	17	$2saa + 27a + 47sa - ssa = 27ss$
17 ÷ 25	18	$aa + \frac{27a}{25} + 27a - \frac{1}{2}sa = 27$
Substitute	19	$2x = \frac{27}{25} + 27 - \frac{1}{2}s = 114$
Then	20	$aa + 2xa = 27 = 7776$
20 C□	21	$aa + 2xa + xx = 27 + xx = 11025$
21 uw 2	22	$a + x = \sqrt{27 + xx} = 105$
22 - x	23	$a = \sqrt{27 + xx} - x = 48$
1 - 23	24	$u = 60 = \text{the Base.}$
per 13	25	$e = \frac{ss}{a} - 2s = 27$
2 - 25	26	$y = 45 = \text{the Cathetus.}$
23 + 25	27	$a + e = 75 = \text{the Hypotenuse.}$

Problem II.

The Difference of the Sides, of any Oblique-angled Plain Triangle; the Difference of the Segments of the Base; And the Difference between the Greater Side and the Base being given; To find the Base, &c.

Let {	1	$d = \text{The Difference of the Sides} = 405.$
	2	$b = \text{The Difference of the Segments} = 495.$
	3	$x = 165 \text{ The Differ. of Greater Side and Base.}$
And	4	$a = \text{the Least Side.}$
Then	5	$d + a + x = \text{the Base}$
And	6	$d + a + x : d + 2a :: d : b$
		Per Theorem 16.
6 ::	7	$db + ba + bx = dd + 2da$
7 ±	8	$2da - ba = db + bx - dd$
8 ÷ 2d - b	9	$a = \frac{db + bx - dd}{2d - b} = \frac{118125}{315} = 375$
1 + 9	10	$d + a = 780 = \text{the Greatest Side.}$
3 + 10	11	$d + a + x = 945 = \text{the Base.}$



Tt

problem

Problem 12.

The Difference of the Sides, of any plain Triangle; the Difference of the Segments of the Base, and the Perpendicular let fall from the Vertical Angle, being given; Thence to find all the Sides.

Let $\left\{ \begin{array}{l} 1 \quad d = 405 \\ 2 \quad b = 495 \end{array} \right\}$ as before.
 And $3 \quad p = 300$
 Quere $4 \quad a = \text{the least Segment.}$



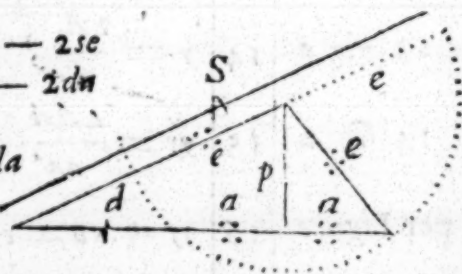
Then $5 \quad b + 2a : d + 2e :: d : b$
 $5 \quad \therefore 6 \quad bb + 2ba = dd + 2de$
 $6 - dd 7 \quad bb - dd + 2ba = 2de$
 Substitute $8 \quad 2x = bb - dd = 81000$
 $7, 8 9 \quad 2x + 2ba = 2de$
 $9 \div 2d 10 \quad \frac{x + ba}{d} = e$
 But $11 \quad pp + aa = ee$ Per Theorem 11.
 $10 \odot 2 12 \quad \frac{xx + 2xba + bbaa}{dd} = ee$
 $11, 12 13 \quad \frac{xx + 2xba + bbaa}{dd} = pp + aa$
 $13 \times dd 14 \quad xx + 2xba + bbaa = ppdd + ddaa$
 $14 \pm 15 \quad bbaa - ddaa + 2xba = ppdd - xx$
 $8, 15 16 \quad 2xaa + 2xba = ppdd - xx$
 $16 \div 2x 17 \quad aa + ba = \frac{ppdd}{2x} - \frac{1}{2}x$
 $17 \text{ C} \square 18 \quad aa + ba + \frac{1}{4}bb = \frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x$
 $18 \text{ uv } 2 19 \quad a + \frac{1}{2}b = \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x}$
 $19 - \frac{1}{2}b 20 \quad a = \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x} - \frac{1}{2}b = 225$
 $20 \times 2 21 \quad 2a = 450$
 $2 + 21 22 \quad b + 2a = 945 = \text{the Base.}$
 $10, \text{Num. } 23 \quad e = 375 = \text{the Lesser Side.}$
 $1 + 23 24 \quad d + e = 780 = \text{the Greater Side.}$

Problem 13.

The Sum of the Two Sides of any plain Triangle, the Difference of the Segments of the Base, and the Perpendicular let fall from the Vertical

Vertical Angle upon the Base, being given; Thence to find the Base and the Sides.

Let	{	1	$s = 1155$	The Sum of the Sides.
		2	$d = 495$	The Difference of the Segments.
		3	$p = 300$	The Perpendicular.
Put	{	4	$a =$	the least Segment.
		5	$e =$	the least Side.
Then		6	$d + 2a =$	the Base.
And		7	$s - 2e =$	the Difference of the Sides.
per Fig.	{	8	$d + 2a : s :: s - 2e : d$	
		9	$aa + pp = ee$	
9 uw 2		10	$\sqrt{aa - pp} = e$	
8 ::		11	$dd + 2da = ss - 2se$	
11 ±		12	$2se = ss - dd - 2da$	
Suppose		13	$2x = ss - dd$	
Then		14	$2se = 2x - 2da$	
14 ∴ 2s		15	$e = \frac{x - da}{s}$	
10, 15		16	$\frac{x - da}{s} = \sqrt{aa - pp}$	
16 ⊙ 2		17	$\frac{xx - 2xda + ddaa}{ss} = aa + pp$	
17 × s		18	$xx - 2xda + ddaa = ssaa + sspp$	
18 ± s		19	$ssaa - ddaa + 2xda = xx - sspp$	
13, 19		20	$2xaa + 2xda = xx - sspp$	
20 ÷ 2x		21	$aa + da = \frac{1}{2}x - \frac{sspp}{2x}$	&c. as before.
21, hence		22	$a = 225$	
22 × 2		23	$2a = 450$	
2 + 23		24	$d + 2a = 945 =$	the Base.
10, Num.		25	$e = 375 =$	the Lesser Side.
1 - 25		26	$s - e = 780 =$	the Greater Side.



Problem 14.

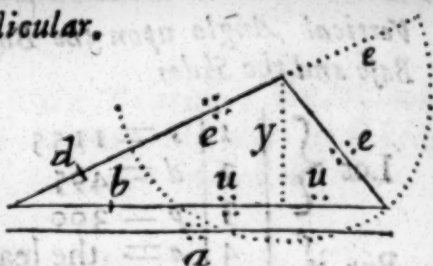
The Area, of any Oblique Angled plain Triang'le; the Difference of the Sides, and the Difference of the Segments of the Base, being given; Thence to find the Base, &c.

Let	{	1	$A = 141750 =$	the Area.
		2	$d = 405$	
		3	$b = 495$	

T t 2

Put

Put {	4	$y = \text{the Perpendicular.}$
Then	5	$a = \text{the Base.}$
	6	$\frac{1}{2}ya = A$
per Figure	7	$a:d+2e::d:b$
7 ∴	8	$ba = dd + 2de$
8 — dd	9	$ba - dd = 2de$
9 ⊙ 2	10	$bbaa - 2ddba + dddd = 4ddee$
per Figure	11	$\frac{a-b}{2} = u$ The Lesser Segment of the Base.
11 ⊙ 2	12	$\frac{aa - 2ba + bb}{4} = uu$
6 × 2	13	$ya = 2A$
13 ÷ a	14	$y = \frac{2A}{a}$
14 ⊙ 2	15	$yy = \frac{4AA}{aa}$
per Figure	16	$yy + uu = ee = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
10 ÷ 4dd	17	$\frac{bbaa - 2ddba + dddd}{4dd} = ee$
16, 17	18	$\frac{bbaa - 2ddba + d^4}{4dd} = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
18 × aa	19	$\frac{bbaa^2 - 2ddba^3 + d^4aa}{4dd} = 4AA + \frac{a^4 - 2ba^3 + bbaa}{4}$
19 × 4dd	20	$\{ bba^4 - 2ddba^3 + d^4aa = 16AAdd + dda^4$
20 ±	21	$bbba^4 - dda^4 + d^4aa - ddbbaa = 16AAdd$
21 ÷	22	$aaaa - ddaa = \frac{16AAdd}{bb - dd}$
22 C□	23	$aaaa - ddaa + \frac{1}{4}dddd = \frac{16AAdd}{bb - dd} + \frac{1}{4}d^4$
23 uw 2	24	$aa - \frac{1}{2}dd = \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}d^4}$
24 + $\frac{1}{2}dd$	25	$aa = \frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}dddd}$
25 uw 2	26	$a = \sqrt{\frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}dddd}} = 945$



Problem 15.

There is an Oblique Angled Plain Triangle, wherein a Perpendicular is let fall from the Vertical Angle upon the Base; the least Side, and the Base are given; And the Rectangle of the Difference of the Sides into the least Side, is Equal to the Square of the Difference of the Segments of the Base. 'Tis required to find the Segments of the Base, &c.

Let $\left\{ \begin{array}{l} 1 \quad c = 56 = \text{the Least Side.} \\ \text{And} \quad 2 \quad B = 92 = \text{the Base.} \\ \text{Put} \quad 3 \quad a + 2c = B \\ \text{Then} \quad 4 \quad y = \text{the Difference of the Sides.} \\ \quad \quad 5 \quad cy = aa \text{ By the Question.} \end{array} \right.$

per Figure $6 \quad B : 2c + y :: y : a, \text{ for } B = a + 2c$

$6 \quad \therefore 7 \quad Ba = 2cy + yy$

$5 \times \frac{8}{2} \quad 8 \quad 2cy = 2aa$

$7 - 8 \quad 9 \quad Ba - 2aa = yy$

$5 \odot \frac{8}{2} \quad 10 \quad ccyy = aaaa$

$10 \div cc \quad 11 \quad yy = \frac{aaaa}{cc}$

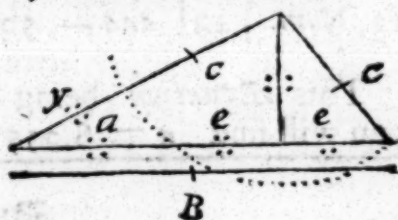
$9, - 11 \quad 12 \quad Ba - 2aa = \frac{aaaa}{cc}$

$12 \times cc \quad 13 \quad ccBa - 2ccaa = aaaa$

$13 \div a \quad 14 \quad ccB - 2cca = aaa$

$14 + 2cca \quad 15 \quad aca + 2cca = ccB$

$15, \text{ in Num.} \quad 16 \quad aaa + 6272a = 288512$



The Value of a , in this Equation, may be found as in the Examples Page 238. viz. by putting $r + e = a$, &c. As in those Examples, you will find $a = 37,55502$ &c.

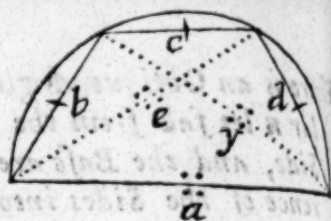
Problem 16.

The Three Chords or Subtenses of Three Arches compleating a Semi-circle, being each given; Thence to find the Diameter of that Circle.

That is, any Trapezium being Inscribed in a Semi-circle, if one of its Sides be the Diameter, and the other Three Sides be given; Thence to find the Diameter or Fourth Side.

Suppose

Let $\left\{ \begin{array}{l} 1 \quad b = 3 \\ 2 \quad c = 4 \\ 3 \quad d = 5 \end{array} \right\}$ the 3 Sides.
 Quere 4 $a =$ the Dia. sought.



Draw the Two Diagonals e , and y
 Then 5 $ca + bd = ey$ Per Theorem 19.
 And $\left\{ \begin{array}{l} 6 \quad aa - bb = yy \\ 7 \quad aa - dd = ee \end{array} \right\}$ Per Theorems 10 and 11.
 5 $\odot$ 2 8 $ccaa + 2bdca + bbdd = eeyy$
 6 $\times$ 7 9 $aaaa - bbba - ddaa + bbdd = eeyy$
 8, = 9 10 $aaaa - bbba - ddaa = ccaa + 2bdca$
 10 $\div a$ 11 $aaa - bba - dda = cca + 2bdc$
 11 $- cca$ 12 $aaa - bba - dda - cca = 2bdc$
 12, Num. 13 $aaa - 50a = 120$

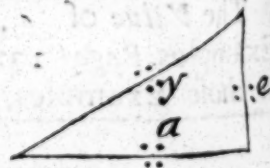
This Equation being Solved as in Example 2. Page 240, you will find $a = 8,05581$ &c.

Problem 17.

In any Right-angled Triangle, the Area, and the Sum of the Hypotenuse when Added to either Side, being given; Thence to find the Sides, &c.

Suppose $\left\{ \begin{array}{l} 1 \quad \frac{ae}{2} = A = 1350 \text{ The Area.} \\ 2 \quad y + e = s = 120 \text{ The Sum, \&c.} \\ 3 \quad \text{Quere } a, e, \text{ and } y \end{array} \right.$

1 $\times$ 2 4 $ae = 2A$
 4 $\div a$ 5 $e = \frac{2A}{a}$
 per Figure 6 $aa + ee = yy$
 2 $- e$ 7 $y = s - e$
 5, 7 8 $y = s - \frac{2A}{a}$
 8 $\odot$ 2 9 $yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$
 5 $\odot$ 2 10 $ee = \frac{4AA}{aa}$
 10 $+ aa$ 11 $aa + ee = \frac{4AA}{aa} + aa$



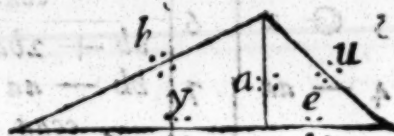
6, 9, 11	12	$\frac{4AA}{aa} + aa = yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$
12, That is	13	$aa = ss - \frac{4sA}{a}$
13 $\times a$	14	$aaa = ssa - 4sA$
14 $\div$	15	$ssa - aaa = 4sA$
15, in Num.	16	$14400a - aaa = 648000$

The Value of a , in this Equation, may be found as in the third Example, Page 241. That is, by making $r + e = a$, &c. it will be found that $a = 60$.

Problem 18.

There is an Oblique-angled Plain Triangle, wherein a Perpendicular is let fall from the Vertical Angle upon the Base; the Sum of each Segment of the Base when Added to its adjacent or next Side, and the Area of the Triangle are given; To find the Perpendicular, and each Side.

Let $\left\{ \begin{array}{l} 1 \ y + b = z = 1500 \\ 2 \ e + u = s = 600 \end{array} \right\}$ Quere y, b, e and u
 And $3 \ A = \text{the Area} = 141750$
 Then $4 \ a = \text{the Perpendicular sought.}$
 $5 \ y + e \times \frac{1}{2}a = A$



$\times 2 \div a$	6	$y + e = \frac{2A}{a}$
er Fig. $\left\{ \right.$	7	$yy + aa = bb$
	8	$ee + aa = uu$
1 $- y$	9	$b = z - y$
2 $- e$	10	$u = s - e$
9 $\odot 2$	11	$bb = zz - 2zy + yy$
10 $\odot 2$	12	$uu = ss - 2se + ee$
7, 11	13	$zz - 2zy = aa$
8, 12	14	$ss - 2se = aa$
13 $\div$	15	$zz - aa = 2zy$
14 $\div$	16	$ss - aa = 2se$
15 $\div 2z$	17	$\frac{zz - aa}{2z} = y$
16 $\div 2s$	18	$\frac{ss - aa}{2s} = e$
17 + 18	19	$\frac{zz - aa}{2z} + \frac{ss - aa}{2s} = y + e$

Having found the Value of a , from the 24th Step, e , and y , will be easily found by these Two Steps. And b, u by the 9th and 10th Step.

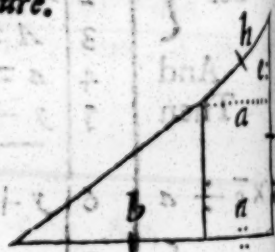
6,	19	20	$\frac{zz - aa}{2z} + \frac{ss - aa}{2s} = \frac{2A}{a}$
20 X 2z		21	$zz - aa + \frac{zss - zaa}{s} = \frac{4zA}{a}$
21 X s		22	$zzs - saa + zss - zaa = \frac{4zAs}{a}$
22 X a		23	$zrsa - saaa + zssa - zaan = 4zAs$
23, Num.		24	$900000a - aaa = 243000000$

Here $a = 300$ found as in the last Prob.

Problem 19.

There is a Right-angled Triangle, wherein a Right Line is drawn Parallel to the Cathetus; there is given, the Cathetus, that Segment of the Hypotenuse next to the Cathetus, and the Alternat Segment of the Base; Thence to find the Base, &c.

Viz. Let	1	$b = 20 \quad c = 24 \quad \text{and} \quad h = 15$
Then	2	$b + a = \text{the Base.} \quad \text{Quere } a$
Here	3	$b + a : c :: a : e \quad \text{Per Figure.}$
And	4	$aa + ea = bh \quad \text{Per Figure.}$
3	5	$\frac{ca}{b + a} = e$
5	6	$\frac{ccaa}{bb + 2ba + aa} = ee$
4	7	$bb - aa = ee$
6,	7	$\frac{ccaa}{bb + 2ba + aa} = bb - aa$
8 X	9	$ccaa = hbbb - bbba + 2hbba - 2ba^3 + hbba$
9	10	$a^4 + 2baaa + ccaa + bbba - hbba - 2hbba = hbb$
That is	11	$aaaa + 40aaa + 751aa - 9000a = 90000$



For a Solution of this Equation, let it be made

$$aaaa + baaa + caa - da = G \quad \text{Viz. } \begin{cases} b = 40. c = 751 \\ d = 9000. G = 90000 \end{cases}$$

Put $r + e = a$

$$\text{Then } \begin{cases} r^4 + 4rrre + 6rree = a^4 \\ brrr + 3brre + 3brce = baaa \\ crr + 2cre + cee = caa \\ -dr - de = -da \end{cases} = G = 90000$$

Let $r = 10$

Then $\left\{ \begin{array}{l} +10000 + 4000e + 6000e \\ +40000 + 12000e + 12000e \\ +75100 + 15020e + 7510e \\ -90000 - 9000e \end{array} \right\} = G = 90000$

That is, $35100 + 22020e + 25510e = 90000$

Hence it will be $22020e + 25510e = 54900$

Consequently, $8,63e + e = 21,52 = D$

And $\left\{ \begin{array}{l} D \\ 8,63 + e \end{array} \right\} = e$

Operation $8,63 \overline{) 21,52} \quad (2,1 = e$
 $+ e = 2,1 \quad 20$

1. Divisor $= 10 \quad 1,52 \quad \text{First } r = 10$

2. Divisor $= 10,7 \quad 1,07 \quad + e = 2,1$

$45 \text{ \&c. } r + e = 12,1 = r \text{ for}$

a second Operation, which being Involved, and Multiplied into the Co-efficients as before; will produce these Numbers.

$\left\{ \begin{array}{l} +21435,8881 + 7086,24e + 878,46e \\ +70862,4400 + 17569,20e + 1452,00e \\ +109953,9100 + 18174,20e + 751,00e \\ -108900,0000 - 9000,00e \end{array} \right\} = G$

Viz. $93352,2381 + 33829,64e + 3081,46e = 90000$

Here because $93352,2381 > 90000$. Therefore $12,1 > a$, and therefore it must be made $r - e = a$, which will produce the same Numbers, only all the second Signs must be changed.

Thus, $93352,2381 - 33829,64e + 3081,46e = 90000$ from whence will arise this Equation,

$+33829,64e - 3081,46e = 3352,2381$

Consequently, $10,9784e - e = 1,08787332 = D$

Operation $10,9784 \overline{) 1,08787332} \quad (0,0999 = e$
 $- e = ,0999 \quad 9792$

1. Divisor $10,88 \quad 108673 \quad \text{Last } r = 12,1$

2. Divisor $10,879 \quad 97911 \quad - e = 0,0999$

3. Divisor $10,8785 \quad 1076232 \quad r - e = 12,0001 = a$

979065

&c.

Problem 20.

In the oblique-angled Triangle CAD , there is given the Side AD , and the Sum of the Sides $AC + CD$; Also within the Triangle there is given the Line AB Perpendicular to the Side CA ; Thence to find the Side CA , &c.

V v

Let

Let { And Then	1	$CA + CD = s = 51$	
	2	$AD = d = 32$	
	3	$AB = b = 21$	
	4	$CA = a$ sought	
	5	$s - a = CD$	
Suppose		the Line DF	
Then		Parallel to AB , and CA produced to F .	
And	6	$\triangle CAB$, and $\triangle CFD$, will be alike.	
But	7	$BC : CA :: DC : CF$	
6, 7	8	$BC = \sqrt{bb + aa}$. Let $AF = e$, and $FD =$	
8	9	$\frac{sa - aa}{\sqrt{bb + aa}} = a + e$	
5	10	$ss - 2sa + aa = \square CD$	
per Fig.	11	$ss - 2sa + aa = aa + 2ae + ee + yy = \square CF + \square FD$	
11 - aa	12	$ss - 2sa = 2ae + ee + yy$	
But	13	$dd = ee + yy = \square AF + \square FD$	
12 - 13	14	$ss - 2sa - dd = 2ae$	
Let	15	$2x = ss - dd$	
14, 15	16	$x - sa = ae$	
16 $\div a$	17	$\frac{x - sa}{a} = e$	
17 + a	18	$\frac{x - sa + aa}{a} = a + e$	
9	19	$\frac{ssaa - 2sada + a^4}{bb + aa} = \square a + e$	
18	20	$\frac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa} = \square a + e$	
18, 19	21	$\left\{ \begin{array}{l} \frac{ssaa - 2sa + a^4}{bb + aa} = \\ = \frac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa} \end{array} \right.$	

This Equation being brought out of the Fractions, and in Numbers, will become,

$$-2018aaaa + 125409aaa - 2464230,25aa + 3546830a - 274183922,25$$

which being Divided by 2018, the Co-efficient of the high Power of a , will

be $\begin{cases} -a^4 + 62,1459 a^3 - 1221,125 aa + 17575,9697 a = \\ = 135869,138875 \text{ \&c.} \end{cases}$

And from hence the *Value* of *a* may be found, as in the *Last Problem*, due regard being had to the *Signs* of every *Term*.

This Work of *Reducing* or preparing *Equations* for a *Solution* by *Division*, hath always been taught both by *Ancient* and *Modern Writers* of *Algebra*; as a Work so necessary to be done, that they do not so much as give a *Hint* at the *Solution* of any *Advised Equation* without it.

Now it very often happens, that in *Dividing* all the *Terms* of an *Equation*, some of their *Quotients* will not only run into a long *Series*, but also into imperfect *Fractions* (as in this *Equation* above) which renders the *Solution* both *Tedious*, and *Imperfect*.

To remedy that *Imperfection*, I shall here shew how this *Equation* (and *Consequently* any other) may be *Resolved* without such *Division* or *Reduction*.

Let $b = 2018$. $c = 125409$. $d = 2464230,25$
 $f = 35468307$. And $G = 274183922,25$

Then the *precedent Equation* will stand thus,

$$-baaaa + caaa - daa + fa = G$$

Put $r + e = a$ as before.

Then will $\begin{cases} -br^4 - 4brre - 6brree = -ba^4 \\ + cr^3 + 3crre + 3cree = +ca^3 \\ -drr - 2dre - dee = -daa \\ + fr + fe \dots = +fa \end{cases} = G$

This is plain and easily Conceived; The next thing will be, how to Estimate the first *Value* of *r*; and to perform that, let *G* be *Divided* by *b*, only so far as to determine how many *Places* of whole *Numbers* there will be in the *Quotient*; *Consequently* how many *Points* there must be (according to the *Height* of the *Equation*.)

Thus $b = 2018$) $G = 274183922,25$ (130000
 $\underline{2018}$
 $7238 \text{ \&c.}$

Now from hence one may as easily guess at the *Value* of *r*, as if all the *Terms* had been *Divided*. That is, I suppose $r = 10$ which being *Involved*, &c. as the *Letters* above *Direct*, will be

$$\begin{array}{r}
 - 20180000 - 8072000e - 12108000e \\
 + 125409000 + 37622700e + 37622700e \\
 - 246423025 - 49284605e - 2464230,25ee \\
 + 354683070 + 35468307e
 \end{array} \} = G$$

$$\text{Viz. } 213489045 + 15734402e + 87239,75ee = 2741839\&c$$

$$\text{Hence } 15734402e + 87239,75ee = 60694877,25$$

$$\text{Consequently, } 180,3e + ee = 695,72 = D$$

$$\text{And } \left\{ \frac{D}{180,3 + e} = e \right.$$

$$\text{Operation } 180,3 \quad 695,72 \quad (3,7 = e$$

$$+ e = 3,7 \quad 549$$

$$1. \text{ Divisor} = 183 \quad 146,72 \quad \text{First } r = 10$$

$$2. \text{ Divisor} = 184,0 \quad 128,80 \quad + e = 3,7$$

&c.

$$r + e = 13,7 = r \text{ for}$$

a second Operation, with which you may proceed, as in the Last Problem; and so on to a Third Operation, if occasion require such Exactness. But this may be sufficient to shew the Method of Resolving any *Adfected Equation* without reducing it; which is not only very exact, but also very ready in Practice, as will fully appear in the last Chapter of this Part, concerning the *Periphery* and *Area* of the *Circle*, &c. wherein you will find a further Improvement in the *Numerical Solution* of *High Equations* than hath hitherto been Published.

CHAP. V.

Practical Problems, and Rules for finding the Superficial Contents or Area's of Right-lin'd Figures.

Before I proceed to the following Problems, it may be convenient to acquaint the Learner, That the Superficies or Area of any Figure, whether it be Right-lin'd or Circular, is composed or made up, of Squares, either Greater or Less, according to the different Measures by which the Dimensions of the Figure are taken or Measured.

That is, if the Demensions are taken in Inches, the Area will be composed of Square Inches: If the Dimensions are taken in Feet, the Area will be composed of Square Feet: If in Yards, the Area will be Square Yards: And if the Dimensions are taken in Poles or Perches (As in Surveying of Land, &c.) then the Area will be Square Perches, &c. These things being understood

Chap. 5. Of *Practical Rules* about *Area's*, &c. 333

and the *Definitions* in the 283 and 284 *Pages* well considered, will help to render the following *Rules* very *Easie*.

Problem 1.

To find the *Superficial Content* or *Area* of a *Square*; Or of any *Right-angled Parallelogram*.

Rule { Multiply the *Length* into its *Breadth*; and the *Product* will be the *Area* required. (See *Lemma 1*. *Page* 302.)

Example. Suppose the *Line* $AB = 6$ *Yards*; And the *Breadth* AC , Or $BD = 3$ *Yards*.

Then $AB \times AC = 6 \times 3 = 18$ will be the *Number* of *Square Yards* contained in the *Area* of the *Parallelogram* $ABCD$.



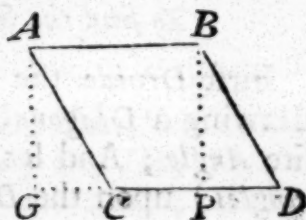
This is so *Evident* by the *Figure* only, that it needs no *Demonstration*.

Problem 2.

To find the *Area* of any *Oblique-angled Parallelogram*, viz. either of a *Rhombus*, Or *Rhomboides*.

Rule { Multiply the *Length* into its *Perpendicular Height* (or *Breadth*); and the *Product* will be the *Area* required.

That is, the *Side* $AB \times BP =$ the *Area* of the *Rhombus* $ABCD$. For if BP be drawn *Perpendicular* to CD , and AG be made *Parallel* to BP ; Then will $GC = PD$, and $GP = CD$. Consequently $\triangle AGC = \triangle BPD$, and $\square ABGP =$ *Rhombus* $ABCD$.



But $AB \times BP = \square ABGP$. Therefore $AB \times BP$; Or $CD \times BP =$ the *Area* of the *Rhombus* $ABCD$,

Example. Suppose the *Side* $AB = 23$ *Inches*, and the *Perpendicular* $BP = 17,5$ *Inches* (being the *shortest* or *nearest Distance* between the two *Sides* AB and CD).

Then $AB \times BP = 23 \times 17,5 = 402,5$ *Square Inches*, being the *Area* of the *Rhombus* required.

The like may be done for any *Rhomboides*, whose *Length* and *Perpendicular Breadth* are given,

Problem

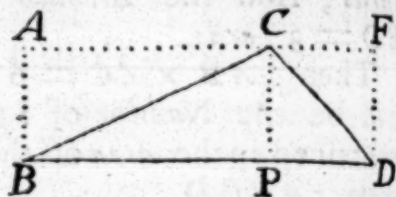
Problem 3.

To find the Superficial Content or Area of any Plain Triangle.

Every Plain Triangle is Equal to half its Circumscribing Parallelogram, (41. e. 1.) which affords the following Rule.

Rule { Multiply the Base of the given Triangle into Half its Perpendicular Height; Or Half the Base into the whole Perpendicular; And the Product will be the Area.

That is, $BD \times \frac{1}{2}CP$, Or $\frac{1}{2}BD \times CP = \text{Area of } \triangle BCD$.
For $AC = BP$, $AB = CP$,
and BC is common to both $\triangle ABC$ and $\triangle BCP$,
therefore $\triangle ABC = \triangle BCP$.
And for the like Reasons $\triangle CFD = \triangle CPD$.



Therefore $\triangle BCP + \triangle CPD = \frac{1}{2}\square ABCD$. Consequently $\frac{1}{2}BD \times CP$. Or $BD \times \frac{1}{2}CP$ will be the Area of the $\triangle BCD$.

Example. Suppose the Base $BD = 32$ Inches, and the Perpendicular Height $CP = 14$ Inches.

Then $\frac{1}{2}BD \times CP = 16 \times 14 = 224$. Or $BD \times \frac{1}{2}CP = 32 \times 7 = 224$.

Or thus, $32 \times 14 = 448$. Then $2) 448$ ($224 = \text{the Area of the Triangle } BCD \text{ in Square Inches}$).

Problem 4.

To find the Superficies or Area of any Trapezium.

First Divide the given Trapezium into Two Triangles, by drawing a Diagonal from one of its Acute Angles to the Opposite Angle; And let fall Two Perpendiculars (from the other Two Angles) upon the Diagonal. As in the following Figure. Then

Rule { Multiply Half the Diagonal into the Sum of the Two Perpendiculars; Or Half the Sum of the Perpendiculars into the Diagonal; And the Product will be the Area.

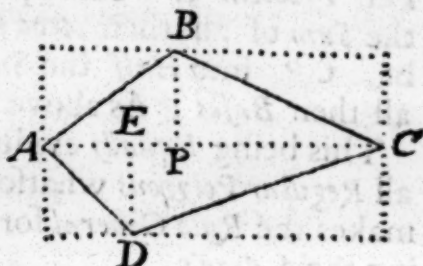
That is, $\frac{1}{2}AC \times BP + ED$. Or $AC \times \frac{1}{2}BP + ED = \text{Area of the Trapezia } ABCD$.

For the $\triangle ABC$ is Half its Circumscribing Parallelogram; And the $\triangle ACD$ is also Half of its Circumscribing Parallelogram; As hath been proved at the Last Problem.

Conse-

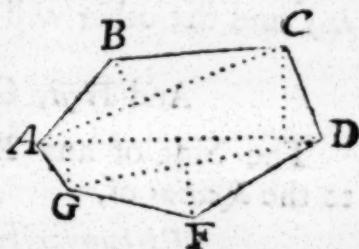
Consequently, $BP + ED \times \frac{1}{2}AC$. Or $\frac{1}{2}BP + \frac{1}{2}ED \times AC$ will be the Area of the Trapezium. As above.

Example. Suppose the Diagonal $AC = 33$ Feet, the Perpendicular $BP = 15$ Feet, and the Perpendicular $ED = 14$ Feet. Then $BP + ED = 29$ Feet; And $BP + ED \times \frac{1}{2}AC = 29 \times 16,5 = 478,5$. Or $AC \times \frac{1}{2}BP + \frac{1}{2}ED = 33 \times 14,5 = 478,5$. Or thus, $29 \times 33 = 957$ Then $2) 957$ (478,5 any of these Products are the Area of the Trapezia $ABCD$.



Problem 5.

To find the Superficial Content or Area of any Irregular Polygon or many Sided Figure; which by some Authors is called a Triangulate, because (as I suppose) it must be Divided into Triangles, as in the Annexed Figure $ABCDFG$; by which it is Evident, that the Sum of the Area's of all those Triangles, found as in the Last Problem, &c. will be the Area of their Circumscribing Polygon.



Problem 6.

To find the Superficies or Area of any Regular Polygon; viz. of any Regular Pentagon, Hexagon, Heptagon, Octagon, &c.

General Rule { Multiply Half the Sum of its Sides, into the Radius of the Inscribed Circle; Or, Half the said Radius, into the Sum of the Sides; And the Product will be the Area required.

That is, $\frac{AB+BD+DE+EF+FG+GH+HK+KA}{2} : \times CP$

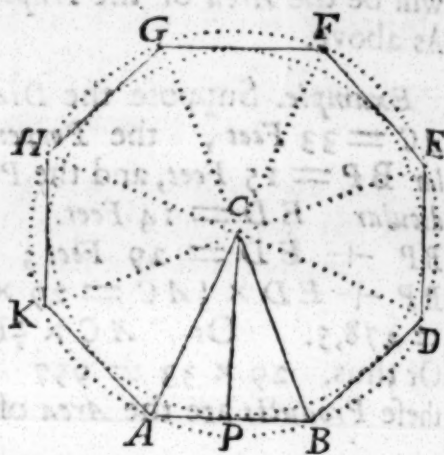
= the Area of the Annexed Octagon; wherein it is Evident, that its Area is composed of so many Equal Isosceles Triangles, as there are Number of Sides in the Polygon, viz. of Eight Isosceles Triangles; whose Bases are the Sides of the Octagon, viz. $AB = BD = DE$, &c. And the Sides of those Triangles, CA, CB, CD , &c. are the Radius's of the Circumscribing Circle; and their Perpendicular Heights, viz. CP , is the Radius of the Inscribed Circle.

But

But the *Area* of any one of those *Triangles*, is $\frac{1}{2}AB \times CP$. Per *Problem 3*. Consequently the *Sum* of all their *Area's* will be, CP into *Half* the *Sum* of all their *Bases*; As above.

This being *Equally* evident in all *Regular Polygons* whatsoever, makes the *Rule General* for finding their *Area's*.

Now because it is required to have the *Radius* of the proposed *Polygon's Inscribed Circle*, I shall here *Insert* (and *Demonstrate*) the *Proportions* that are between the *Sides* of several *Regular Polygons* and the *Radius's*, both of their *Inscribed* and *Circumscribing Circles*; the one will help to *Delimitate* or *Project* the *Polygon* (if *Occasion* require it) and the other will help to find its *Area*.



And *First*, Of an *Equilateral Triangle*.

The *Side* of any *Equilateral Plain Triangle*, is in *Proportion* to the *Radius* of,

its $\begin{cases} \text{Circumscribing Circle, As } 1 : \text{To } 0,57735027 \text{ \&c.} \\ \text{Inscribed Circle, As } 1 : \text{To } 0,28867513 \text{ \&c.} \end{cases}$

And to its *Perpendicular Height*, As $1 : \text{To } 0,86602540 \text{ \&c.}$

That is, $\begin{cases} AB : CD :: 1 : 0,57735027 \\ AB : CG :: 1 : 0,28867513 \end{cases}$

And $AB : AG :: 1 : 0,86602540$

Demonstration.

Let $AB = BD = 1$.

Then will $BG = GD = 0,5$

But $\square AB - \square BG = \square AG$.

Per *Theorem 11*.

That is, $1 - 0,25 = 0,75 = \square AG$:

Consequently, $\sqrt{0,75} = 0,86602540 = AG$.

Then $AG : AB :: AB : AH$. Per *Theorem 13*.

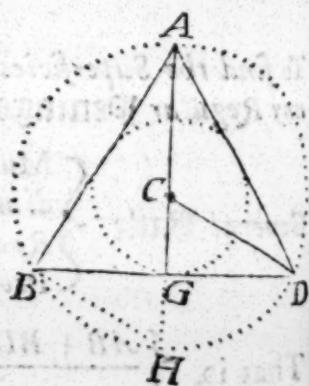
That is, $0,8660254 : 1 :: 1 : 1,15470054 \text{ \&c.} = AH$

Then $\frac{1}{2}AH = 0,57735027 = AC$. Again, $AG : DG :: DG : CG$

That is, $0,8660254 : 0,5 :: 0,5 : 0,28867513 = CG$. Q. E. D.

Now by the help of the *First* of these *Proportions*, it will be *Easie* to *Resolve* the following *Problem*.

Problem



Chap. 5. Of Practical Rules about Area's, &c. 337

problem 7.

The Side of any Equilateral Plain Triangle being given; To find its Area.

Example. Suppose the Side of the proposed Triangle ABC to be 25 Inches, viz. $AB = BC = CA = 25$

First $1 : 0,8660254 :: AB = 25 : 21,650635$

$= BP$. Per Theorem 13.

Then $AP (= \frac{1}{2}CA) \times BP =$ the Area of $\triangle ABC$. Per Rule to Problem 3.

That is, $12,5 \times 21,650635 = 270,6329$ the Area in Square Inches.

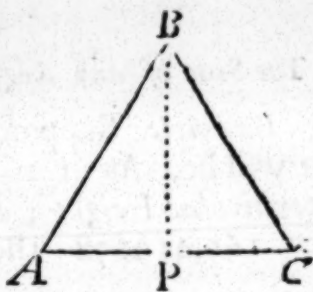
Or this Problem may be otherwise Resolved, thus.

Let $b = AP = \frac{1}{2}AC$. Then $2b = AB$.

But $\square AB - \square AP = \square BP$. Per Theorem 11.

That is, $4bb - bb = 3bb = \square BP$. Consequently, $\sqrt{3bb} = BP$

Then $b\sqrt{3bb} = BP \times \frac{1}{2}AC$. Viz. $\sqrt{3bbbb} =$ the Area of the Triangle.



Secondly, For a Pentagon.

The Side of any Regular Pentagon, is in Proportion to the Radius of

its { Circumscribing Circle, As 1 : To 0,85065080 &c.
Inscribed Circle, As 1 : To 0,68819096 &c.

And to its Perpendicular Height, As 1 : To 1,53884176 &c.

Viz. { $AB : AC :: 1 : 0,85065080$
 $AB : CH :: 1 : 0,68819096$
 $AB : AH :: 1 : 1,53884176$

Demonstration.

Let $AB = 1$. And draw the Diagonals AD , AF and DG , which will be Equal to one another. Then will $AG \times DF : + AD \times GF = AF \times DG$ Per Theorem 19.

Consequently, $AG \times DF = AF \times DG : - AD \times GF$.

That is, $\square AB = \square AD : - AD \times GF = 1$.

Hence it will be $AD = 1,61803398$

Then $\square AD - \square DH = \square AH$ Per Theor. 11. But $DH = \frac{1}{2}AB$

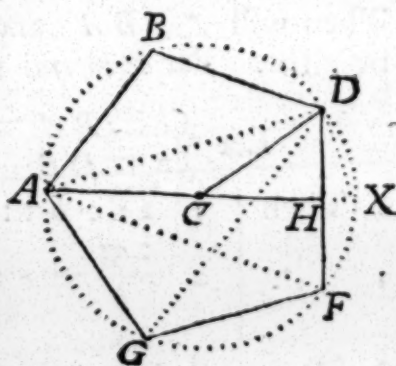
Therefore $\sqrt{\square AD - \frac{1}{4}\square AB} = AH = 1,53884176$

Again $AH : AD :: AD : AX = 2AC$. For $\triangle AHD$

and $\triangle ADX$ are alike.

X x

Ergo



Irgo $\frac{\square AD}{AH} = 2AC = 1,70130161$. Hence $AC = 0,85065080$
 But $AH - AC = CH = 0,68819096$ &c. Q. E. D.

From hence it will be *Easie* to *Resolve* the following *Problem*.

Problem 8.

The Side of any Regular Pentagon being given ; To find its Area.

Example. Suppose the given *Side* to be 15 Inches long ; then it will be, As 1 : 1,53884176 :: 15 : 22,0826264 the *Perpendicular Height* ; And by the general *Rule* $22,0826264 \times \frac{15}{2} = 165,619698$ the *Area* required.

Thirdly, For an Octagon.

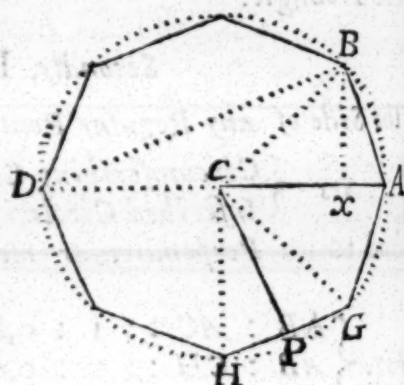
The Side of any Regular Octagon, is in Proportion to the Radius of its { *Circumscribing Circle, As 1 : To 1,30656296 &c.*
Inscribed Circle, As 1 : To 1,20710678 &c.

Viz. { $BA : CA :: 1 : 1,30656296$
 $BA : CP :: 1 : 1,20710678$

Demonstration.

Draw the *Right Line* DB and from the *Point* B , let fall the *Perpendicular* Bx upon the *Diameter* DA .

Then will $\triangle DBA$ and $\triangle Dx B$ be alike. Per *Theorems* 10 and 12.



Let { $b = BA = 1$. $a = CA$
 $e = DB$ and $y = Bx$

Then 1 $2a : b :: e : y$ Viz. $DA : BA :: DB : Bx$

1 $\therefore$ 2 $\frac{2ay}{b} = e = DB$

2 $\odot$ 2 3 $\frac{4aay}{bb} = ee = \square DB$

But 4 $4aa - \frac{4aay}{bb} = bb$.

That is, $\square DA - \square BB = \square BA$. Per *Theorem* 11.

4 $\times bb$ 5 $4bbaa - 4aay = bbbb$

Again 6 { $\frac{1}{2}aa = yy$. For $Cx = Bx$
 and $\square Cx + \square Bx = \square CB = aa$

5,	6	7	$4bbaa - 2a^4 = b^4$. Or $2a^4 - 4bbaa = -b^4$
7	$\div 2$	8	$aaaa - 2bbaa = -\frac{1}{2}b^4$
8	C□	9	$a^4 - 2bbaa + b^4 = b^4 - \frac{1}{2}b^4 = \frac{1}{2}b^4$
9	uw 2	10	$aa - bb = \sqrt{\frac{1}{2}b^4}$
10	+- bb	11	$aa = bb + \sqrt{\frac{1}{2}b^4}$
11	uw 2	12	$a = \sqrt{bb + \sqrt{\frac{1}{2}b^4}} = 1,30656296 \text{ \&c.} = CA$
Then		13	$aa - \frac{1}{4}bb = \square CP$. Viz. $\square CH - \square HP = \square CP$
13	uw 2	14	$\sqrt{aa - \frac{1}{4}bb} = 1,20710678 \text{ \&c.} = CP$

From hence it will be Easie to find the *Area* of any *Octagon*.

Problem 9.

The *Side* of any *Regular Octagon* being given; To find its *Area*.

Example. Suppose the *Side* given to be 12 Inches long; First, As 1 : 1,20710678 :: 12 : 14,48528136 = the *Radius* of its *Inscribed Circle*. Then $12 \times 4 = 48$ is Half the *Sum* of its *Sides*; And $48 \times 14,48528136 = 695,2935$ the *Area* required?

Fourthly, For a *Decagon*.

The *Side* of any *Regular Decagon*, (viz. a *Polygon* of Ten equal *Sides*) is in *Proportion* to the *Radius* of,

Its { *Circumscribing Circle*, As 1 : To 1,61803398 &c.
Inscribed Circle, As 1 : To 1,53884176 &c.

Viz. { $BA : CA :: 1 : 1,61803398$
 $BA : CP :: 1 : 1,53884166$

Demonstration.

Let { $b = BA = 1$. $a = CA$
 $e = DB$. and $y = Bx$

Then 1. $2a : b :: e : y$
 That is, $DA : BA :: DB : Bx$

2. { $2ay = be$
 and $2y = \frac{be}{a}$

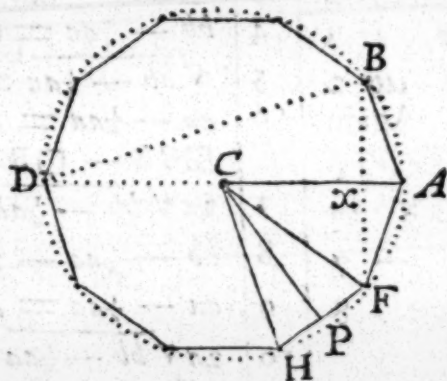
But 3. $2y : e :: 1 : 1,61803398$ Vide *Pentagon*.

4. $\frac{1e}{1,61803398} = 2y = \frac{be}{a} = \frac{1e}{a}$

5. $1,61803398 = a = CA$

Again 6. { $aa - \frac{1}{4}bb = \square CP$.
 Viz. $\square CF - \square PF = \square CP$. Per *Theor. 11.*

That is, 7. $\sqrt{2,61803396 - 0,25} = 1,53884176 = CP$



X x 2

problem

Confectary.

Hence if the *Side* of any *Regular Dodecagon* be given, the *Radius* of its *Inscribed Circle* may be *Easily* obtained, and thence the *Area* found, as in the *Last Problem*.

The *Work* of the foregoing *Polygons* being well considered, will help the young *Geometer* to *Raise* the like *Proportions* for others, if his *Curiosity* or *Occasion* requires them: And not only so, but they will also help to form a true *Idea* of a *Circle's Periphery* and *Area*, according to the *Method* which I shall lay down in the next *Chapter* for finding them both.

C H A P. VI.

A new and Easie Method of finding the Circle's Periphery and Area, to any assigned Exactness (or Number of Figures;) by one Equation only. Also a new and facile Way of making Natural Sines, and Tangents.

Let us suppose (*what is very Easie to Conceive*) the *Circle's Area* to be composed or made up, of a *vast Number* of *Plain Isosceles Triangles*, having their *Acutest Angles* all meeting in the *Circle's Center*: And let us *imagine* the *Bases* of those *Triangles* so very small, that their *Sides* and their *Perpendicular Heights*, viz. the *Radius's* of their *Circumscribed* and *Inscribed Circles* (*vide Problem 6.*) may become so very near in *Length* to each other, as that they may be taken one for another, without any *sensible Error*. Then will the *Peripheries* of their *Circumscribing* and *Inscribed Circles*, become (*although not co-incident, yet*) so very near to each other, as that either of them may be indifferently taken for one and the same *Circle*.

But how to find out the *Sides* of a *Polygon* (*viz. the Bases of those Isosceles Triangles*) to such a convenient *Smallness*, as may be necessary to determin and settle the *Proportion* betwixt a *Circle's Diameter* and its *Periphery* (*to any assigned Exactness*) hath hitherto been a *Work* which required great *Care* and much *Time* in its *Performance*, as may easily be conceived from the *Nature* of the *Method* used by all those who have made any considerable *Progress* in it, viz. *Archimedes*, *Snellius*, *Hugenius*, *Méius*, *Van Culen*, &c. These proceeded with the *Bisecting* of
an

an Arch, and found the Value of its Chord to a convenient Number of Figures at every single Bisection, repeating their Operations until they had Approached to the Chord designed.

And this Method is made Choice of by the Learned Dr. Wallis, in his Treatise of Algebra; wherein, after he hath given us a large Account of the different Enquiries made by several (very Eminent in Mathematical Sciences) in order to find out some easier and more expeditious Way of Approaching to the Circle's Periphery, as in Chap. 82, 84, 85, 86, and several other Places, he comes to this Result (Page 321.)

" 'Tis true (saith he) we might in like manner proceed by
 " continual Trisection, Quinquisection, or other Section, if
 " we had for these as convenient Methods of Operation, as we
 " have for Bisection: But because Euclide shews how to Bi-
 " sect an Arch Geometrically, but not to Trisect, &c. and the
 " one may be done (Algebraically) by resolving a Quadratick
 " Equation, but not those other, without Equations of a high-
 " er Composition: I therefore make Choice of a continual Bi-
 " section. &c.

And then he lays down these following Canons

The Subtense of $\frac{1}{6}$

		1	into 6
of $\frac{1}{12}$	$\sqrt{2} - \sqrt{3}$		into 12
of $\frac{1}{24}$	$\sqrt{2} - \sqrt{2 + \sqrt{3}}$		&c. 24
of $\frac{1}{48}$	$\sqrt{2} - \sqrt{2 + \sqrt{2 + \sqrt{3}}}$		48
of $\frac{1}{96}$	$\sqrt{2} - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}$		96
&c.	$\sqrt{2} - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}$		192
	$\sqrt{2} - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}}$		384
	$\sqrt{2} - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}}}$		768
	&c.		&c.

How tedious and troublesome the Work of these complicated Extractions is, I leave to the Consideration of those, who either have had Experience therein, or out of Curiosity will give themselves the Trouble of making Trial.

Again in Page 347, the Doctor Inserts a particular Method proposed by Libnitius, Published in the *Acta Eruditorum* at Lipsich, for the Month of February, 1682. in order to find the Circle's Area, and consequently its Periphery, which is this,

As 1 : To $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}$ &c.
 Infinitely :: So is the Square of the Diameter : To the Circle's Area. But this convergeth so very slowly, that it is not worth the time to pursue it.

I shall here propose a new Method of my own, whereby the Circle's Periphery (and consequently its Area) may be obtained infinitely

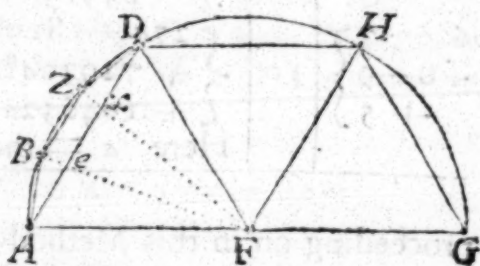
infinitely near the Truth, with much greater Ease and Expedition, than either that of *Bisection*, or that of *Libnitius*, as above; or any other Method that I have yet seen, it being performed by *Resolving* only one *Æquation*, deduced by an *Easie Process* from the Property of a Circle (*known to every Cooper*) which is this,

The Radius of every Circle, is Equal to the Chord of one Sixth Part of its Periphery.

That is, $AD = DH = HG$, the Chords of $\frac{1}{3}$ Part of the Semi-circle; are each Equal to AF its Radius.

Then if the Arch AD be Trisected, it will be $AB = BZ = ZD$.

Let $\begin{cases} R = AF = 1 \\ c = AD = 1 \\ a = AB \end{cases}$ Quere a .



Then	1	$R : a :: a : \frac{aa}{R} = Be$
And	2	$R : a :: R - \frac{aa}{R} : c - 2a$
That is, For	3	$FB : BZ :: Fe : ex = AD - 2a$ $\triangle AFB$, and $\triangle BAe$, are alike. And $AB = Ae = Dx$, &c.
2	4	$Rc - 2Ra = Ra - \frac{aaa}{R}$
4 x &c.	5	$3Ra - aaa = R Rc$. That is, $3a - aaa = 1$ Here a = the Chord of $\frac{1}{18}$ Part of the Circle. For $\frac{1}{3}$ of $\frac{1}{6} = \frac{1}{18}$.

Next, To Trisect the Arch AB .

Let	1	$3y - y^3 = a$ the Last Chord.
1 ⊙ 3	2	$27y^3 - 27y^5 + 9y^7 - y^9 = a^3$
1 x 3	3	$9y - 3y^3 = 3a$
3 - 2	4	$9y - 30y^3 + 27y^5 - 9y^7 + y^9 = 3a - a^3 = 1$ Here y = the Chord of $\frac{1}{24}$ Part of the Circle.

Again, To Trisect the Arch, whereof y is the Chord.

Let	1	$3a - a^3 = y$
⊙ 3	2	$27a^3 - 27a^5 + 9a^7 - a^9 = y^3$
⊙ 5	3	$243a^5 - 405a^7 + 270a^9 - 90a^{11} + 15a^{13} - a^{15} = y^5$
		1 ⊙ 7

1 ⊙ 7	4	$\{ 2187a^7 - 5103a^9 - 5103a^{11} - 2835a^{13}$
		$\{ + 945a^{15} = y^7$
1 ⊙ 9	5	$\{ 19683a^9 - 59049a^{11} - 78732a^{13} -$
		$\{ - 61236a^{15} = y^9$
1 × $\frac{9}{9}$	6	$271 - 9a^3 = 9y$
2 × $\frac{30}{30}$	7	$810a^3 - 810a^5 - 270a^7 - 30a^9 = 30y^3$
3 × $\frac{27}{27}$	8	$\{ 6561a^5 - 10935a^7 - 7290a^9 - 2430a^{11}$
		$\{ + 405a^{13} - 27a^{15} = 27y^5$
4 × $\frac{9}{9}$	9	$\{ 19683a^7 - 45927a^9 + 45927a^{11} -$
		$\{ - 25515a^{13} + 8505a^{15} = 9y^7$
6 — 7 } + 8 — 9 } + 5 }	10	$\{ 27a - 819a^3 + 7371a^5 - 30888a^7$
		$\{ + 72930a^9 - 107406a^{11} +$
		$\{ + 104652a^{13} - 69768a^{15} \} =$
		Here $a =$ the Chord of $\frac{1}{62}$ Part of the Circle.

proceeding on in this Method of continually *Trisecting* the *Arch* of every new *Chord*, and still *Connecting* the produced *Equations* into one, as in the *Two last Trisections*; it will not be difficult to obtain the *Chord* of any assigned *Arch*, how small soever it be.

Now in order to facilitate the Work of *Raising* these *Equations* to any considerable Height; it will be convenient to add a few useful *Observations*, concerning the Nature, and of such *Contractions* as may be safely made in them; which being well understood, will render the Work very *Easie*.

1. I have observed that every *Trisection* will gain or advance one *Figure* in the *Circles* periphery; but no more. Therefore so many *places* of *Figures*, as are at first designed to be perfect in the *Periphery*; so many *Trisections* must be repeated to *Raise* an *Equation*, that will produce a *Chord* answerable to that *Design*.

2. I have also found, that all the *Superiour Powers* (of a) whose *Indices* are greater than the *Number* of *Trisections*. (viz. whose *Indices* are greater than the *Number* of designed *Figures*) may be wholly *Rejected* as *Insignificant*.

3. When once the *Number* of *Trisections*, and thence the highest *Power* (of a) is *Determin'd*, the third *Process* (viz. the third *Trisection*) may be made a fixt or a constant *Canon*; for by it, and *Multiplication* only, all the succeeding *Trisections* (how many soever they are) may be compleated, without repeating the several *Involutions*.

4. In *Raising* and *Collecting* the *Co-efficients*, of the several *Powers* (of a) it will be sufficient to retain only so many significant *Figures*.

Figures (at a^3) as there is design'd to be places of Figures in the Periphery: (or at most but two more) And every succeeding superiour Power may be allowed to decrease two places of significant Figures. But herein great care must be taken to supply the Places of those Figures that are omitted, with Cyphers; that so the whole and exact Number of places may be truly adjusted; otherwise all the Work will be Erronious;

Now the Number of those supplying Cyphers, may be very conveniently denoted by Figures placed within a Parenthesis; thus $576(8)a^3$, may signifie $57600000000a^3$, as in the following Equations. The like may be done with Decimal Parts, thus $(,7)658$ may signifie $,0000000658$ &c. which will be found very useful in the Solution of these, and the like Equations.

The aforesaid Contractions may be safely made, because both the Superiour Powers of a , which are Rejected; as also those Numbers that are omitted in the Co-efficients (and supplied with Cyphers) would produce Figures so very remote from Unity, as that they would not affect the Chord Designed. That is, they would not affect the Chord in that place wherein the Designed Periphery is concerned. As will in part appear in the following Example.

If these Directions be carefully minded, it will be easie to Raise an Equation that will produce the Side of a Regular Polygon; whose Number of Sides shall be vastly Numerous, Consequently infinitely small. But I presume it will be sufficient for an Example, to find the Side of a Polygon consisting of 258280326 equal Sides: That is, if I find the Chord of $\frac{1}{258280326}$ part of the Circle's Periphery, and that requires but Sixteen Trisections, which being ordered as before directed will produce this Equation.

$$\left. \begin{aligned} &43046721a - 332360179486968612(4)a^3 \\ &+ 769837653199714(20)a^5 - 8491218532841(35)a^7 \\ &+ 54633331143(50)a^9 - 230083348(66)a^{11} \\ &+ 6830988(79)a^{13} - 15072(94)a^{15} \end{aligned} \right\} = 1$$

Here the Value of a , will have 23 places of Figures true. This is, the Sides of the Inscribed and Circumscribed Polygons will be exactly the same to 23 places of Decimal Parts, but not further; all which may be easily obtain'd at Two Operations. And for the first, it will be sufficient to take only Three Terms of the Equation; which will admit of being yet further Contracted, thus,

Yy

Let

$$\text{Let } \left\{ \begin{array}{l} 43046721a - 3323601794(12)a^3 \\ + 76983765(27)a^5 \end{array} \right\} = 1$$

And let $r + e = a$. Then Rejecting all the Powers of e , that arise by Involution above eee ,

$$\text{it will be } r^3 + 3rre + 3ree + eee = aaa$$

$$\text{And } r^5 + 5r^4e + 10r^3ee + 10rree = a^5.$$

Then the first single Value of r , may be thus found

$$43046721) 1,000000000 (,000000002 = r$$

This $,000000002 = r$ being duely Involved and its Powers Multiplied into their respective Co-efficients, will produce

$$\left. \begin{array}{l} + ,86092442 + 43046721e \\ - ,02658881 - 3988322e - 199416(9)ee - 3324(18)eee \\ + ,00024635 + 61587e + 6159(9)ee + 308(18)eee \end{array} \right\} = 1$$

$$\text{Viz } ,83459196 + 39119986e - 193257(9)ee - 3016(18)eee = 1$$

$$\text{Hence } 39119986e - 193257(9)ee - 3016(18)eee = 0,165408$$

All the Terms of this last Equation being Divided by 193257(9) the Co-efficient of ee , it will then become

$$,0000002024e - ee - 156(5)eee = ,00000000000000008558968 = 1$$

$$\text{Consequently, } \left\{ \frac{D + 156(5)eee}{,0000002024 - e} = e \right.$$

Operation.

$$\begin{array}{r} ,0000002024) ,0000000000000000008558968 (,000000004 \\ - e = ,0000000043: + ,00000000000000000009984 = 156(5)eee \\ 1 \text{ Di. } ,000000198) ,0000000000000000008568952 (0000000043 \\ 2 \text{ Di. } ,0000001981 \end{array}$$

792

6489

5943

5465

3962

&c.

$$\text{First } r = ,00000002$$

$$+ e = ,000000004327$$

$$r + e = ,000000024327 = a.$$

second Operation.

Or rather New r for

Now if this first Value of $a = ,000000024327$ were not continued to more Places of Figures, by a second Operation, but Multiplied into the Number of Chords.

$$\text{Viz. } ,000000024327 \times 258280326 = 6,28318539 \text{ \&c.}$$

Periphery of that Circle whose Diameter is 2, nearer than ei

Archime

Archimedes or Mærius's Proportions: For Archimedes makes it 6,285714 &c. viz. As 7 To 22 And Mærius makes it 6,28318584 &c. viz. As 113 To 355.

But if the whole Equation before proposed, be now taken, and we proceed to a second Operation; the Value of a may be Increased with Twelve Places of Figures more; and those may be obtain'd by plain Division only.

Thus, let $r + e = a$, as before; and let all the Powers of e be now rejected as Insignificant.

Then will $\left\{ \begin{array}{l} r + e = a \\ r^3 + 3r^2e = a^3 \\ r^5 + 5r^4e = a^5 \\ r^7 + 7r^6e = a^7 \end{array} \right\}$ And $\left\{ \begin{array}{l} r^9 + 9r^8e = a^9 \\ r^{11} + 11r^{10}e = a^{11} \\ r^{13} + 13r^{12}e = a^{13} \\ r^{15} + 15r^{14}e = a^{15} \end{array} \right\}$

The several Powers of $r = ,000000024327$ being Raised, and Multiplied into their respective Co-efficients, will produce these following Numbers.

+ 1,047197581767	+ 43046721e	} = 1
— ,047849196598394865	— 5900751e	
+ ,000655906484595355	+ 134810e	
— ,000004281440413375	— 1232e	
+ ,000000016302517863	+ 6e	
— ,000000000040631167	— 0e	
+ ,000000000000071388	+ 0e	
— ,000000000000000093	— 0e	

Viz. $1,000000026474745106 + 37279554e = 1$

Hence $37279554e = - ,000000026474745106 = D$

Or rather $- 37279554e = ,000000026474745106 = D$

Consequently, $\left\{ \frac{D}{37279554} = -e \right.$

Operation.

$37279554) ,000000026474745106 ((,15)710167967 = -e$

260956878

37905730

37279554

62617660

37279554

&c.

(viz. $\sqrt{2} + \sqrt{3}$) must have 99 places of Figures in it. The third Root (viz. $\sqrt{2} + \sqrt{2} + \sqrt{3}$) must have 96 places in it, &c. every Extraction being allowed to Decrease three Places, that so the last Root (viz. the Chord sought) may consist of 24 Places of Figures, as above.

I say, who ever duly considers the trouble of these so often repeated Extractions, will, I presume, be pleased with what I have done. For truly when I consider of the great Time and Care required in them, I cannot but admire at the Patience of the Laborious Van Culen, who proceeded that way until he had found the Circle's Periphery to Thirty Six Places of Figures, to wit, 6,28318530717958647692528676655900576.

These Numbers are said to be Ingraven upon his Tomb-stone in St. Peter's Church in Leyden, for a Memorial of so great a Work.

Having thus obtained the Circle's Periphery; its Area may easily be found (to the same Number of Figures) by Problem 6.

That is, If Half the Periphery of any Circle be Multiplied into Half its Diameter, the Product will be that Circle's Area, as will appear further on. Therefore 3,141592653589793 will be the Area of the Circle whose Diameter is 2.

Thus have I shewed the young Geometer how to find the Circle's Periphery and Area to what Exactness he pleases to approach; for precisely true they cannot be found, notwithstanding the late Pretensions of a certain Frenchman, who hath Published to the World (in the Works of the Learned) that after twenty five Years Study he had found the Quadrature of the Circle: But if he had perused the 83 Chapter of Dr. Wallis's Algebra, he might there have seen his Error, viz. the Impossibility of what he pretended to; For it is as impossible to Square the Circle (that is, to find its true Area) as it is to find the Root of a Surd Number.

Note, What I have here proposed and done by the Trisection of an Arch, may as easily, and much more speedily be perform'd by Quinisection, or Septisection, &c. But because the Scheme for Trisection is more Simple, and may be easier understood by a Learner than those of the other Sections (of which see my Compendium of Algebra Pages 76 and 79.) I have for that Reason made choice of Trisection.

As to the Proportion of one Circle to another; And of the Circle to the Ellipsis, &c. those shall be fully shewed when we come to the fifth Part,

Before

Before I conclude this Part, I shall make some Use or Application of the above found Periphery, in finding the Quantity of Angles; which is done by the help of Right Lines, called Sines and Tangents, the Length whereof are Calculated to every Degree and Minute of a Quadrant, by much Labour. But I shall here shew how to find the Natural Sine (and consequently the Natural Tangent) of any proposed Arch or Angle, by Two Equations, without the help of any precedent Sine as usual; which I did some Years ago communicate to the Ingenious Mr. Joseph Raphson, and he so well approved of them, as to make them the 20 and 21 Problems in the Second Edition of his *Analysis Aëquationum Universalis*.

And because in finding the Quantity of Angles, every Circle is supposed to be Divided into 360 equal Parts, called Degrees; every Degree is Sub-divided into 60 Parts, called Minutes; And every Minute into 60 Seconds, &c. (vide Page 294.)

Therefore 360) 6,2831853 &c. (0,0174532925 &c. is an Arch of the abovefound Periphery, equal to the Arch of one Degree.

And 60) 0,0174532925 &c. (0,0002908882 &c. = the Arch of one Minute.

Then if the given Arch (or Angle) be Less than 45 Degrees, Reduce it into Minutes, and Multiply those Minutes into this constant Multiplier, viz. 0,0002908882 calling the Product p . And for the Sine sought put a . Then will

$$-aaaa + 12paaa - 195aa - 36ppaa + 240pa = 45pp.$$

Example.

Let it be required to find the Sine of $19^{\circ} . 13' = 1153'$.

Here $0,0002908882 \times 1153 = 0,3353940946 = p$.

And $-a^4 + 4,024729a^3 - 199,049611aa + 80,494583a = 5,06201394$.

Let $r + e = a$

$$\text{Then } \begin{cases} rr + 2re + ee = aa \\ rrr + 3rre + 3ree = aaa \\ rrrr + 4rrre + 6rree = aaaa \end{cases}$$

Note, In this Case the First r , may always be taken Equal to the First Figure in the Product $= p$. Viz. here $r = 0,3$ which being Involved as its Powers direct, and those Powers Multiplied into the respective Co-efficients of the Equation, it will be

$$\left\{ \begin{array}{l} + 24,1483 + 80,49e \\ - 17,9144 - 119,43e - 199,05ee \\ + 0,1086 + 1,08e + 3,62ee \\ - 0,0081 - 0,11e - 0,54ee \end{array} \right\} = 5,06201394$$

$$\text{Viz. } 6,3344 - 37,97e - 195,97ee = 5,06201 \quad \text{Hence}$$

Hence $37,97e + 195,97ee = 1,27239$

And $,193e + ee = ,006492 = D$

Theorem $\left\{ \frac{D}{,193 + e} = e \right.$

Operation. $,193) ,006492 (,029 = e$

$+ e = ,029$ 42

1. Divisor $,21$ 2292

2. Divisor $,222$ 1998

First $r = ,3$

$+ e = ,029$

$r + e = ,329 = r$ for a second Operation.

Which being Involved and Multiplied &c. as before, will produce these Numbers.

$+ 26,48271781 + 80,49458e$

$- 21,54532894 - 130,97464e - 199,0496ee$

$+ 0,14332578 + 1,30692e + 3,9724ee$

$- 0,01171611 - 0,14244e - 0,6494ee$

$Viz. 5,06899854 - 49,31558e - 195,7266ee = 5,06201394$

Hence $49,31558e + 195,7266ee = ,0069846$ which being Divided by $195,7266$ the Co-efficient of ee will become

$,25196e + ee = ,0000356854 = D$

Then $\left\{ \frac{D}{,25196 + e} = e \right.$

Operation. $,25196) ,0000356854 (,0001415 = e$

$+ e = ,00014$ 2520

1. Divisor $,2520$ 104854

2. Divisor $,25210$ 100840

40140

25210

&c.

Last $r = ,329$

$+ e = ,0001415$

$r + e = a = ,3291415$ being the Natural Sine of $19^\circ 13'$.

As was required.

Thus you may find the Right Sine of any Arch or Angle, Less than 45 Degrees.

But

But if the given *Arch* be Greater than 45 Degrees, you must take its Complement to 90°. viz. Subtract it from 90 Degrees; and Reduce the Remainder into Minutes, as before.

Then Multiply the Square of those Minutes into this constant Multiplier, 0,000000084616 calling their Product p ; and putting $a =$ the Sine sought, as before. Then will

$$a_4 + 28a^3 + 195aa + 36paa + 108pa - 28a = 196 - 81p$$

Example.

Suppose it were required to find the Sine of 75°. 32'. Or (which is the same thing) to find the Co-sine of 14°. 28'. = 868'. whose Square $753424 \times 0,000000084616 = 0,06375172518 = p$. Hence the Equation in Numbers will be,

$$aaaa + 28aaa + 197,295062aa - 21,114814a = 190,8361102588$$

$$\text{Let } r - e = a \quad \text{And } r = 1$$

$$\text{Then } \begin{cases} rr - 2re + ee = aa \\ rrr - 3rre + 3ree = aaa \\ rrrr - 4rrre + 6rree = aaaa \end{cases}$$

Note, I here take $r = 1$ because the Arch is so near to 90°, and therefore I make it $r - e = a$.

$$\text{Then } \left\{ \begin{array}{l} - 21,1148 + 21,11e \\ + 197,2956 - 394,59e + 197,29ee \\ + 28,0000 - 84,00e + 84,00ee \\ + 1,0000 - 4,00e + 6,00ee \end{array} \right\} = 190,8361$$

$$\text{Viz. } 205,1808 - 461,48e + 287,29ee = 190,8361$$

$$\text{Hence } 461,48e - 287,29ee = 14,3447$$

$$\text{And } 1,606e - ee = ,049930 = D$$

$$\text{Theorem } \left\{ \frac{D}{1,606 - e} = e \right.$$

$$\text{Operation } 1,606 \overline{) ,049930} \quad (,031 = e$$

$$\quad - e = ,031 \quad 471$$

$$1. \text{ Divisor } \underline{1,57} \quad 2830$$

$$2. \text{ Divisor } \underline{1,575} \quad 1575$$

&c.

$$\text{First } r = 1,000$$

$$- e = ,031$$

$$r - e = 0,269 = r$$

for a Second Operation; which being Involved as before will produce these following Numbers.

= 20

$$\begin{array}{rcl}
 - 20,460254766 & + & 21,11481e \\
 + 185,252368710 & - & 382,35783e & + 197,2951ee \\
 + 25,475889852 & - & 78,87272e & + 81,5960ee \\
 + 0,881647759 & - & 3,63941e & + 5,6337ee
 \end{array}$$

$$\begin{array}{rcl}
 \text{Viz. } 191,149651515 & - & 443,75515e & + 284,5248ee = \\
 = 190,836110259
 \end{array}$$

Hence it will be $443,75515e - 284,5248ee = 0,313541256$

$$\text{And } 1,55963e - ee = ,0011019821 = D$$

$$\text{Then } \left\{ \frac{D}{1,55963 - e} = e \right.$$

$$\begin{array}{rcl}
 \text{Operation. } 1,55963 &) & ,0011019821 \quad (,0007088 = e \\
 - e = ,00070 & & 109123
 \end{array}$$

$$\begin{array}{rcl}
 1. \text{ Divisor } 1,5589 & & 1075210 \\
 2. \text{ Divisor } 1,55893 & & 935358 \\
 & & 1398520 \\
 & & 1247144
 \end{array}$$

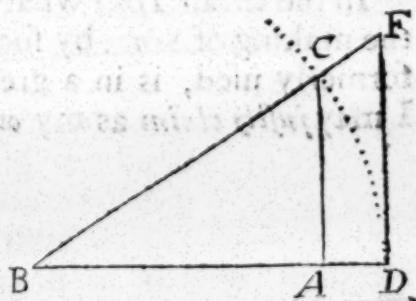
$$\begin{array}{rcl}
 \text{Last } r = ,969 & & \&c. \\
 - e = ,0007068
 \end{array}$$

$$r - e = a = ,9682932 \quad \text{the Sine of } 75^\circ. 32'. \text{ as was required.}$$

Having found the *Sine* and *Co-sine* of any *Arch*; the *Tangent* is usually found by this *Proportion*.

Viz. $\left\{ \begin{array}{l} \text{As the Co-sine of any Arch : Is to the Sine of that Arch ::} \\ \text{So is the Radius : To the Tangent of the same Arch.} \end{array} \right.$

For supposing $BC = BD$ Radius, AC the *Sine* of the *Arch* CD . Then BA is the *Co-sine*, and FD the *Tangent* of the same *Arch*.
But, $BA : CA :: BD : FD$, &c.



Now by this *Proportion* there is required to be given, both the *Sine* and *Co-sine* of the same *Arch*, to find the *Tangent*.

'Tis true, if the *Radius*, and either the *Sine* or the *Co-sine* be given, the other may be found
Thus, $\sqrt{BC^2 - CA^2} = BA$. Or $\sqrt{BC^2 - BA^2} = CA$.

But if either the *Sine* or *Co-sine* be given, the *Tangent* may (I presume) be more easily found by the following *Theorems*.

Let $BC = 1$. $CA = S$. $BA = x$. and $FD = T$.
Then if S be given, T may be found by this

$$\text{Theorem } \left\{ \sqrt{\frac{SS}{1-SS}} = T \right.$$

Or if x be given, T may be found by this

$$\text{Theorem } \left\{ \sqrt{\frac{xx}{1-xx}} = T \right.$$

Example.

Let the *Sine* of $19^\circ. 13'$. (*before found*) be given,
viz. ,3291415 = S . To find T the *Tangent* of the same *Arch*.

First ,3291415 $\times$,3291415 = 0,108334127 = SS .

Again $1 - 0,108334127 = ,891665873 = 1 - SS$.

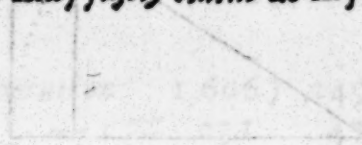
Then ,891665873), 108334127 (,1214963253

And $\sqrt{,1214963253} = ,3485632 = T$, the *Tangent*
of $19^\circ. 13'$. as was required.

And so you may proceed to find T = the *Tangent*, when
 x = the *Co-sine* is given.

Perhaps it may here be expected that I should have shewed and
Demonstrated (or at least have *Inserted*) the *Proportions* from
whence the foregoing *Equations* for making *Sines* were pro-
duced. But I have *Omitted* that, as also their *Use* in *Computing*
the *Sides* and *Angles* of *Plain Triangles* by the *Pen* only, (*viz.*
without the help of Tables) for the *Subject* of another *Discourse*
hereafter, if *Health* and *Time* permit.

In the mean *Time* what is here done may suffice to shew, that
the making of *Sines* by such a *Laborious* and *Operous* Way, as was
formerly used, is in a great *Measure* overcome; which I think
I may justly claim as my own.



A N

INTRODUCTION TO THE Mathematicks.

PART IV.

CHAPTER. I.

Definitions of a Cone, and its Sections.

There are several *Definitions* given of a *Cone*: The Learned Doctor *Barrow* upon *Euclid* hath it thus.

“ A *Cone*, (*saith he*) is a Figure made, when one Side of a
“ *Rectangle Triangle* (*viz.* one of those Sides that contain the
“ *Rectangle Triangle*) remaining fixed, the *Triangle* is turn'd
“ round about, till it return to the Place from whence it first
“ moved. And if the fixed *Right Line* be Equal to the other,
“ which containeth the *Rectangle*; then the *Cone* is a *Rect-*
“ angled *Cone*: But if it be less, it is an *Obruse Angled Cone*;
“ if greater an *Acute Angled Cone*. The *Axis* of a *Cone* is
“ that fixed *Line* about which the *Triangle* is moved. The
“ *Base* of a *Cone* is the *Circle*, which is describ'd by the *Right*
“ *Line* moved about. (*Defi.* 18, 19, 20 *Euclid.* 11)

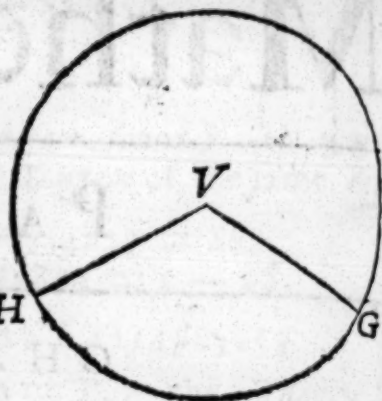
Sir Jonas Moor in his *Treatise of Conical Sections* (taken out of the Works of *Mydorgius*) Defines it thus.

“ If a *Line* of such a length as shall be needful, shall upon
“ a *Point* fixed above the *Plain* of a *Circle*, so move about the
“ *Circle* until it return to the *Point* from whence the motion be-
“ gun; the *Surperficiēs* that is made by such a *Line*, is called a
“ *Conical Superficiēs*; and the *Solid Figure* contained within
“ that *Superficiēs* and the *Circle*, is called a *Cone*. The *Point*
“ remaining still is the *Vertex* of the *Cone*. &c.

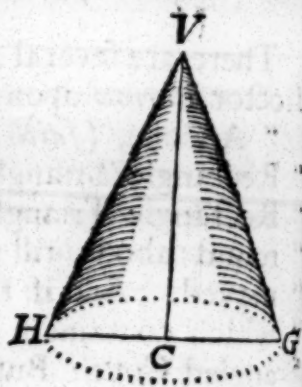
Altho' both these *Definitions* are equally true, and with a little Consideration may be pretty easily understood; Yet I shall here propose one, very different from either of them; and as I presume more plain and Intelligible, especially to a Learner.

If a Circle describ'd upon stiff Paper (or any other pliable Matter) of what bigness you please, be cut into Two, Three or more *Sectors*; either equal or unequal, and one of those *Sectors* be so roll'd up as that the *Radius's* may exactly meet each other, it will Form a *Conical Superficies*.

That is, if the *Sector* HVG be cut out of the Circle, and so roll'd up as that the *Radius's* VH and VG may just meet each other in all their parts, it will form a *Cone*; and the Center V , will become a *Solid Point*, called the *Vertex* of the *Cone*, the *Radius* VH being every where equal, will be the *Side* of the *Cone*; and the *Arch* HG will become a *Circle*, whose Area is called the *Cone's Base*.

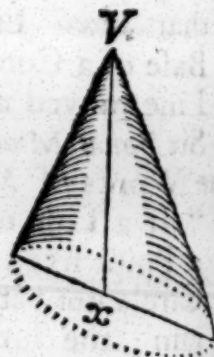


A *Right Line* being supposed to pass from the *Vertex* or point V , to the Center of the *Cone's Base*, as at C , that Line (*viz.* VC) will be the *Axis*, or *Perpendicular Height* of the *Cone*.



If a *Solid* be exactly made in such a Form, it will be a compleat or perfect *Cone*; which I shall all along call a *Right Cone*, because its *Axis* VC stands at *Right-angles* with the plain of its *Base* HG ; and its *Sides* are every where equal.

Any *Cone* whose *Axis* is not at *Right-angles* with the Plain of its *Base*, may be properly called an *imperfect Cone*; because its *Sides* are not every where equal (as in the *Annexed Figure*). Now such an *imperfect Cone*, is usually called a *Scaleneum* or *Oblique Cone*.



Any *Solid Cone* may be cut by *Plains*, (which I shall hereafter call *Right-lines*) into *Five Sections*:

See

Se^t. 1.

If a Right Cone be cut directly thorough its Axis, the Plain or Superficies of that Section, will be a plain Isosceles Triangle, as HVG Figure 2. viz. the Sides (HV and VG) of the Cone will be the Sides of the Triangle; the Diameter (HG) of the Cones Base, will be the Base of the Triangle; And (VC) its Axis, will be the Perpendicular Height of the Triangle.

Se^t. 2.

If a Right Cone be cut (any where) off by a Right Line parallel to its Base, as hg. (it will be easie to conceive, that) the Plain of that Section will be a Circle, because the Cone's Base is such; where in one thing ought to be clearly understood, which may be laid down as a Lemma to Demonstrate the Properties of the following Sections.

Lemma { If any two Right Lines Inscribed within a Circle, do Cut or Cross each other (as hg doth bb in the Annexed Figure) the Rectangle made of the Segments of one of the Lines, will be Equal to the Rectangle made of the Segments of the other Line. (vid. Theorem 15. Page 309.)

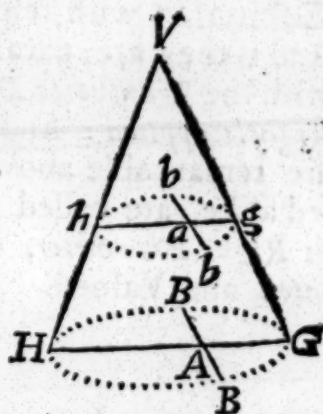
That is, $ha \times ga = ba \times ab$ } &c.
And $HA \times GA = BA \times AB$ }

Consequently if $ba = ab$.

And if $BA = AB$.

Then it will be, $ha \times ga = \square ba$.

And in the Cone's Base $HA \times GA = \square BA$



Se^t. 3.

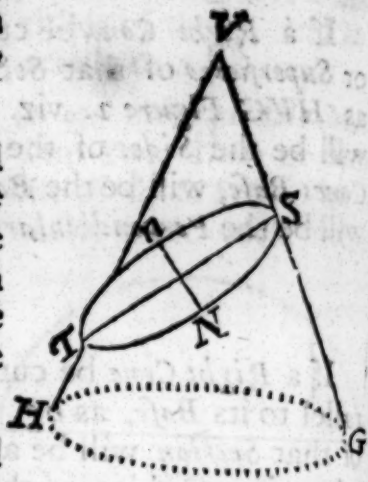
If a Right Cone be (any where) cut off by a Right Line, that cuts both its Sides, but not Parallel to its Base (as TS in the following Figure) the Plain of that Section will be an Ellipsis (vulgarly called an Oval) viz. an Oblong or imperfect Circle, which hath several Diameters, and Two particular Centers. That is,

1. Any Right Line that Divides an Ellipsis into Two equal Parts is called a Diameter; amongst which, the Longest, and the Shortest are particularly distinguished from the rest, as being of most General Use: The other are only Applicable to particular Cases.

2. The

2. The Longest Diameter (as TS) is called the *Transverse Diameter*, or *Transverse Axis*; being that *Right Line* which is drawn thorough the *Middle* of the *Ellipsis*, and doth shew or Limit its *Length*.

3. The Shortest Diameter, called the *Conjugate Diameter*, is a *Right Line* that doth intersect or cross the *Transverse Diameter* at *Right Angles*, in the *Middle* or *Common Center* of the *Ellipsis*; (as Nn) and doth Limit the *Ellipsis Breadth*.



4. The Two Points, which I call particular *Centers* of an *Ellipsis*, (for a Reason which shall be shew'd further on) are two Points in the *Transverse Diameter*, at an equal distance each way from the *Conjugate Diameter*; and are usually called *Foci*, *Focus's*, or *Burning Points*.

5. All *Right Lines* within the *Ellipsis* that are parallel to one another, and can be *Divided* into Two equal *Parts*; are called *Ordinates* with respect to that *Diameter* which *Divides* them. And if they are parallel to the *Conjugate*, viz. at *Right Angles* with the *Transverse Diameter*; then they are called *Ordinates Rightly applied*: And those Two that pass through the *Focusses* are remarkable above the rest, which being equal, and Situated alike, are called both by one Name: viz. *Latus Rectum*, or *Right Parameter*, by which all the other *Ordinates* are *Regulated* and *Valued*. As will appear further on.

Sect. 4.

If any *Cone* be cut into Two *Parts*, by a *Right Line* parallel to one of its *Sides*, (as SA in the following Scheme) the *Plain* of that *Section* (viz. $SbBABbS$) is called a *Parabola*.

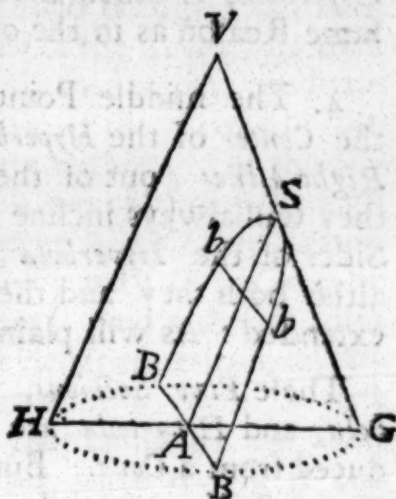
1. A *Right Line* being drawn through the middle of any *Parabola*, (As SA) is called its *Axis*, or intercepted *Diameter*.

2. All *Right Lines* that intersect or cut the *Axis* at *Right Angles*, (As BB , and bb are supposed to Cut or Cross SA) are called *Ordinates Rightly applied* (as in the *Ellipsis*) and the *Greatest Ordinate*, as BB , which *Limits* the *Length* of the *Parabola Axis* (SA) is usually called the *Base* of the *Parabola*.

3. The

3. That *Ordinate* which passes through the *Focus* or *Burning Point* of the *Parabola*, is called the *Latus Rectum*, or *Right Parameter* (As in the *Ellipsis*) because by it, all the other *Ordinates* are *Proportioned*; and may be found.

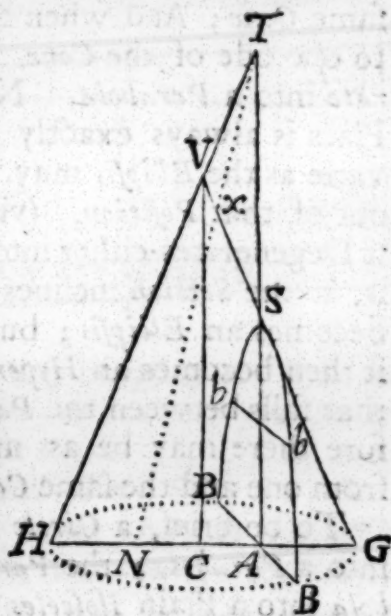
4. The *Node*, *Focus*, or *Burning Point* of the *Parabola*, is a point in its *Axis* (but not a *Center* as in the *Ellipsis*) distant from the *Vertex* or top of the *Section* (viz. from *S*) just one fourth Part of the *Latus Rectum*. As shall be shewed further on.



5. All *Right Lines* drawn within a *Parabola* parallel to its *Axis*, are called *Diameters*; and every *Right Line* that any of those *Diameters* doth *Bisect* or cut into two equal Parts, is said to be an *Ordinate* to that *Diameter* which *Bisects* it.

Sect. 5.

If a *Cone* be any where cut by a *Right Line*, either parallel to its *Axis*, (as *S A*. Or otherwise as *x N*.) So as the cutting *Line* being continued through one *Side* of the *Cone*, (as at *S*, or *x*) will meet with the other *Side* of the *Cone*, if it be continued or produced beyond the *Vertex V*, as at *T*. Then is the *Plain* of that *Section* (viz. the *Figure S b B b S*) is called an *Hyperbola*.



1. A *Right Line* being drawn through the middle of any *Hyperbola*; viz. within the *Section*, (as *S A*, or *x N*) is called the *Axis* or intercepted *Diameter*, (As in the *Parabola*) and that part of it which is continued or produced out of the *Section*, until it meet with the other *Side* of the *Cone* continued; viz. *Ts*, or *Tx*, &c. Is called the *Transverse Diameter*; or *Transverse Axis* of the *Hyperbola*.

2. All *Right Lines* that are drawn within an *Hyperbola*, at *Right Angles* to its *Axis*, are called *Ordinates Rightly applied*. As in the *Ellipsis* and *Parabola*.

3. That

3. That *Ordinate* which passes through the *Focus* of the *Hyperbola*, is called *Latus Rectum* or *Right Parameter*, for the same Reason as in the other *Sections*.

4. The middle Point of the *Transverse Diameter*, is called the *Center* of the *Hyperbola*, from whence may be drawn Two *Right Lines* (out of the *Section*) called *Asymptotes*, because they will always incline (*that is, come nearer and nearer*) to both Sides of the *Hyperbola* but never meet with (or touch) them; altho' both they and the Sides of the *Hyperbola* were *Infinitely* extended: As will plainly appear in its proper Place.

These *Five Sections*, viz. the *Triangle*, *Circle*, *Ellipsis*, *Parabola*, and *Hyperbola* are all the *Plains* that can possibly be produced from a *Cone*. But of them the *Three* last are only called *Conick Sections*, both by the *Ancient* and *Modern Geometers*.

Scholium.

Besides the foregoing *Definitions*, it may not be amiss to add how one *Section* may (or rather doth) *Change* or *Degenerate* into another.

An *Ellipsis* being that plain of any *Section* of the *Cone*, which is between the *Circle* and *Parabola*, it will be easie to conceive that there may be great *Variety* of *Ellipses* produced from the same *Cone*; And when the *Section* comes to be exactly parallel to one side of the *Cone*, then doth the *Ellipsis* *Change* or *Degenerate* into a *Parabola*. Now a *Parabola* being that *Section* whose Plain is always exactly *Parallel* to the side of the *Cone*, cannot varie as the *Ellipsis* may. For so soon as ever it begins to move out of that *Position*, (viz. from being *Parallel* to the *Cone's Side*) it *Degenerates* either into an *Ellipsis*, or into an *Hyperbola*. That is, if the *Section* inclines towards the Plain of the *Cone's Base*, it becomes an *Ellipsis*; but if it incline towards the *Cone's Vertex*, it then becomes an *Hyperbola*, which is the Plain of any *Section* that falls between the *Parabola* and the *Triangle*. And therefore there may be as many *varieties* of *Hyperbolas* produced from one and the same *Cone*, as there may be *Ellipses*.

To be brief, a *Circle* may change into an *Ellipsis*; the *Ellipsis* into a *Parabola*; the *Parabola* into an *Hyperbola*, and the *Hyperbola* into a Plain *Isoceles Triangle*. And the *Center* of the *Circle*, which is its *Focus* or *Burning Point*, doth as it were part or divide it self into two *Focus's* so soon as ever the *Circle* begins to *Degenerate* into an *Ellipsis*; but when the *Ellipsis* changes into a *Parabola*, one end of it flies open, and one of its *Focus's* vanishes

and the remaining *Focus* goes along with the *Parabola* when it degenerates into an *Hyperbola*. And when the *Hyperbola* degenerates into a *Plain Isoscles Triangle* this *Focus* becomes the *Vertical Point* of the *Triangle*. (viz. the *Vertex* of the *Cone*). So that the *Center* of the *Cone's Base*, may be truly said to pass gradually through all the *Sections*, until it arrive at the *Vertex* of the *Cone*; still carrying its *Latus Rectum* along with it. For the *Diameter* of a *Circle* being that *Right Line* which passes through its *Center* or *Focus*, and by which all other *Right Lines* drawn within the *Circle* are regulated and valued; may (*I presume*) be properly called the *Circle's Latus Rectum*; And altho' it loses the Name of *Diameter* when the *Circle* degenerates into an *Ellipsis*, yet it retains the Name of *Latus Rectum*, with its first Properties in all the *Sections*, gradually shortening as the *Focus* carries it along from one *Section* to another, until at last it and the *Focus* become *Coincident*, and *Terminate* in the *Vertex* of the *Cone*.

I have been more particular, and fuller in these *Definitions*, than is usual in *Books* of this Subject, which I hope is no Fault, but will prove of Use especially to a *Learner*; and altho' they may perhaps seem a little strange and at first hard to be understood; yet when they are well considered and compared with a *Cone*, Cut into such *Sections* as have been *Defined*; they will not only be found true, but will also help to form a true and clear *Idea* of each *Section*.

CHAP. II.

Concerning the chief Properties of the Ellipsis.

Note, If the *Transverse Diameter* of any *Ellipsis*, as *TS* in the following Figure, be *Intersected* or *Divided* into any two *Parts* by an *Ordinate* *Rightly Applied*; As at the *Points A, C, a, &c.* Then are those *Parts TA, TC, Ta, and SA, SC, Sa, &c.* usually called *Abscissæ's* (which signifies *Lines* or *Parts cut off*) and by the *Rectangle* of any two *Abscissæ's*, is meant the *Rectangle* of such two *Parts*, as being *Added together*, will be *Equal* to the *Transverse Diameter*.

$$\text{As } TA + SA = TS. \text{ And } TC + SC = TS.$$

$$\text{Or } Ta + Sa = TS, \&c.$$

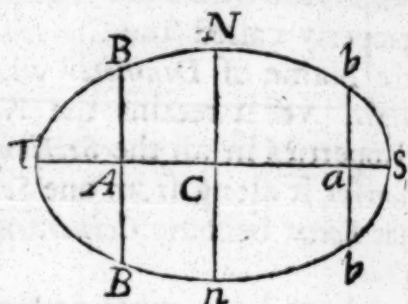
Section 1.

Every *Ellipsis* is proportioned, and all such Lines as relate to it are *Regulated*, by help of one general *Theorem*.

Theorem. { *As the Rectangle of any two Abscissa's: Is to the Square of half the Ordinate which Divides them :: So is the Rectangle of any other two Abscissa's: To the Square of half that Ordinate which Divides them*

That is,

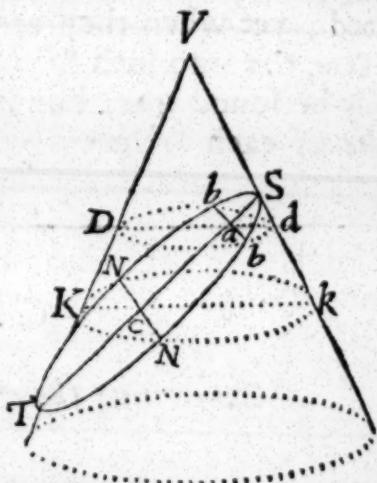
$$\begin{aligned} TA \times SA : \square BA :: Ta \times Sa : \square ba \\ TA \times SA : \square BA :: TC \times SC : \square NC \\ TC \times SC : \square NC :: Ta \times Sa : \square ba \\ \text{\&c.} \end{aligned}$$



Demonstration.

Let the Annexed Figure represent a *Right Cone* cut through both its *Sides*, by the *Right Line* TS, then will the *Plain* of that *Section* be an *Ellipsis*, (per *Sect. 3. Chap. 1.*) TS will be the *Transverse Diameter* NCN and *bab* will be *Ordinates Rightly Applied*. As before.

Again, if the Lines *Dd* and *Kk* be parallel to the *Cone's Base*, they will be *Diameters* of *Circles* (per *Sect. 2. Chap. 1.*) Then will $\triangle TCK$ and $\triangle TaD$ be alike. Also $\triangle Sad$ and $\triangle SCK$ will be alike.



Ergo	1	$Sa : ad :: SC : Ck$	} Per Theorem 13.
And	2	$TC : CK :: Ta : aD$	
1 ∴	3	$Sa \times Ck = ad \times SC$	
2 ∴	4	$Ta \times CK = TC \times aD$	
2 × 3	5	$Sa \times Ck \times Ta \times CK = ad \times SC \times TC \times aD$	per Axiom 3.
But	6	$CK \times Ck = \square NC$	} Per Lemma Sect. 2.
And	7	$aD \times ad = \square ba$	
Then		for $CK \times Ck$, and $aD \times ad$. Take $\square NC$, and $\square ba$	
5, 6, 7,	8	$Sa \times Ta \times \square NC = TC \times SC \times \square ba$	per Axiom 5.
Hence	9	$Sa \times Ta : \square ba :: TC \times SC : \square NC$	vide Pag. 194.

Q. E. D.

Or the Truth of these Proportions may be otherwise proved by a Circle, without the help of the Cone; Thus.

Let any Ellipsis be Circumscrib'd, and Inscrib'd with Circles, as in the following Figure. Then from any Point in the Circumscrib'd Circle's Periphery, as at B, draw the Right Line Ba, Parallel to the Semi-conjugate Diameter NC, then will ba be a Semi-Ordinate Rightly Applied to the Transverse Diameter TS. As before.

Again, from the Point b, (in the Ellipsis Periphery) draw the Right Line bd, Parallel to the Transverse TS. And draw the Radius BC.

Then will $\triangle BCa$ and $\triangle Cfd$ be alike

Therefore 1 $\left\{ \begin{array}{l} BC : Ba :: Cf : dC \\ \text{Per Theorem 13.} \end{array} \right.$

But 2 $\left\{ \begin{array}{l} TC = BC, NC = Cf \\ \text{and } ba = dC \end{array} \right.$

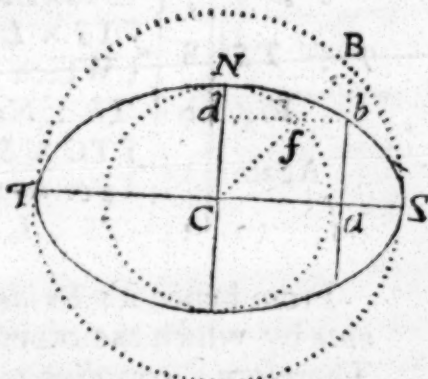
Consequ. 3 $TC : Ba :: NC : ba$

Or 4 $TC : NC :: Ba : ba$

4 in $\square$'s 5 $\square TC : \square NC :: \square Ba : \square ba$

But 6 $\left\{ \begin{array}{l} Ta \times Sa = \square Ba \\ \text{Per Lem. Sect. 2. Chap. 1.} \end{array} \right.$

Therefore 7 $\left\{ \begin{array}{l} Ta \times Sa : \square ba :: TC \times SC = \square TC : \square NC. \\ \text{As before.} \end{array} \right.$



And so for any other Abscissæ's and their Semi-ordinate.

These Proportions being found to be the true, and common Properties of every Ellipsis, all that is further required in (or about) that Section, may be easily deduced from them.

Sect. 2. To find the Latus Rectum, Or Right Parameter of any Ellipsis.

There are several ways of finding the Latus Rectum; but I think none so easie, and shews it so plainly to be the Third Principal Line in the Ellipsis, as the following Theorem.

Theorem $\left\{ \begin{array}{l} \text{As the Transverse Diameter : Is in Proportion to the} \\ \text{Conjugate} :: \text{So is the Conjugate : To the Latus} \\ \text{Rectum.} \end{array} \right.$

Fix. (in the following Figure) $TS : Nn :: Nn : LR$ the Latus Rectum.

Demonstration.

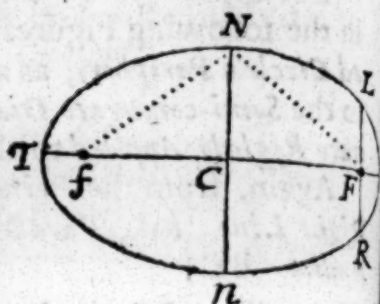
From the last Proportions take either of the Antecedents, and Consequent, viz. either $TC \times SC : \square NC$. Or $Ta \times Sa : \square ba$

A a a 2

and

and make TS the *Third Term*, to which find a *fourth Proportional*, and it will be $= LR$:

Thus	1	$TC \times SC : \square NC :: TS : LR$
But	2	$\begin{cases} TC = SC \\ \text{And } NC = Cn \end{cases}$
Therefore	3	$TC \times SC = \frac{1}{4} \square TS$
And	4	$\square NC = \frac{1}{4} \square Nn$
1, 3, 4	5	$\frac{1}{4} \square TS : \frac{1}{4} \square Nn :: TS : LR$
5	6	$\frac{1}{4} \square TS \times LR = \frac{1}{4} \square Nn \times TS$
$6 \times \frac{4}{4}$	7	$\square TS \times LR = \square Nn \times TS$
$7 \div TS$	8	$\begin{cases} TS \times LR = \square Nn \\ \text{Which gives the following Analogy.} \end{cases}$
Viz.	9	$TS : Nn :: Nn : LR$
Again	10	$\begin{cases} TC \times SC : \square NC :: Ta \times Sa : \square ba. \\ \text{Per common Properties.} \end{cases}$
1, 10	11	$TS : LR :: Ta \times Sa : \square ba$



From hence it's *Evident*, that LR thus found, is that *Ordinate* by which the other *Ordinates* may be *Regulated* and found. Therefore (according to its *Definition Sect. 3. Chap. 1.*) it is the true *Latus Rectum*. Q. E. D.

Confectary.

Hence it follows, that if the *Transverse* and *Conjugate Diameters* of any *Ellipsis* are given (either in *Lines* or *Numbers*) the *Latus Rectum* may be easily found; And then any *Ordinate*, whose *Distance* from the *Conjugate* is given, may be found, As above.

Sect. 3. To find the Focus of any Ellipsis.

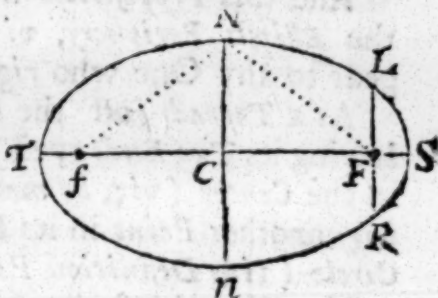
The *Focus* is the *Distance* of the *Latus Rectum* from the *Conjugate* or *Middle* of the *Ellipsis* (vide *Definition 4. Page 358.*) And that *Distance* is always a *Mean Proportional* between the *Half Sum*, and *Half Difference* of the *Transverse* and *Conjugate Diameters*, which gives this *Theorem*.

Theorem $\left\{ \begin{array}{l} \text{From the Square of Half the Transverse, Subtract} \\ \text{the Square of Half the Conjugate; the Square Root of} \\ \text{their Difference, will be the Distance of each Focus} \\ \text{from the Middle, or common Center of the Ellipsis.} \end{array} \right.$

That is, supposing the *Points* f , and F , to be the *Two Focuses* viz. $fC = CF$, and $TC = \frac{1}{2}TS$. $NC = \frac{1}{2}Nn$. Then $TC + NC : fC :: fC : TC - NC$. Ergo $\square fC = \square TC - \square NC$. Consequently, $fC = \sqrt{\square TC - \square NC}$.

Demonstration.

First	1	$TS \times LR = \square Nn$.	Per 8 Step of the last Process.
And	2	$TS : LR :: TF \times SF : \square LF$.	Per common Properties.
That is,	3	$TS : LR :: TC + CF \times TC - CF : \frac{1}{4} \square LR = \square LF$	
3	4	$\left\{ \begin{array}{l} \frac{1}{4} \square LR \times TS = \\ \square TC - \square CF \times LR \end{array} \right.$	
4 $\div LR$	5	$\frac{1}{4} LR \times TS = \square TC - \square CF$	
1 $\div \frac{1}{4}$	6	$\frac{1}{4} TS \times LR = \frac{1}{4} \square Nn = \square NC$	
5,	7	$\square NC = \square TC - \square CF$.	
7, $\pm$	8	$\square CF = \square TC - \square NC$	
8 uw 2	9	$CF = \sqrt{\square TC - \square NC}$	



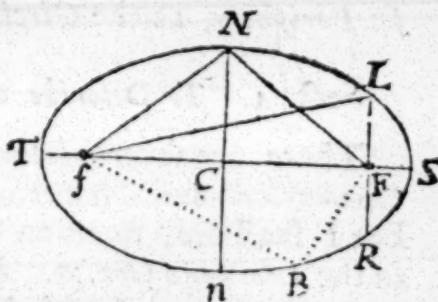
Now from hence is deduced that Notable Proposition, upon which is grounded the usual Method of Describing an Ellipsis and Drawing of Tangents, &c.

Proposition { If from the Two Focus's of any Ellipsis, there be drawn Two Right Lines, so as to meet each other in any Point of the Ellipsis Periphery; the Sum of those Lines will be Equal to the Transverse.

Viz. $fN + NF = TS$. $fL + LF = TS$. Or $fB + BF = TS$ &c.

Demonstration.

First	1	$\left\{ \begin{array}{l} \square CF + \square NC = \square TC \\ \text{Per 8th of the Last.} \end{array} \right.$	
But	2	$\left\{ \begin{array}{l} \square CF + \square NC = \square NF \\ \text{Per Theorem II.} \end{array} \right.$	
1,	3	$\square NF = \square TC$	
2	4	$NF = TC$	
3 uw 2	5	$2NF = 2TC = TS$	
Hence	6	$TS : LR :: TF \times FS : \square LF$.	Per common Properties
Again	7	$\frac{1}{2} TS : \frac{1}{2} LR :: TF \times FS : \square LF$	
Conseq.	8	$\frac{1}{2} TS = TC$. And $\frac{1}{2} LR = LF$	
But	9	$TC : LF :: TC + CF \times TC - CF : \square LF$	
Ergo	10	$TC \times LF = \square TC - \square CF$	
7	11	$\square fF + \square LF = \square fL$ Per Theorem II.	
But	12	$4 \square CF + \square LF = \square fL$ for $2CF = fF$	
That is,	13	$4 \square TC \times LF = 4 \square TC - 4 \square CF$	
8 $\times$ 4	14	$4 \square TC + \square LF = 4 \square TC \times LF : + \square fL$	
10 + 11	15	$4 \square TC : - 4 \square TC \times LF : + \square LF = \square fL$	


 13 uw 2

13	uv	2	14	$2TC - LF = fL$	But $2TC = TS$
14	$+ LF$	15	$2TC = fL + LF$		
Ergo		16	$fL + Lf = TS$		

Q. E. D.

And this *Proposition* must needs hold true to every *Point* in the *Ellipsis Periphery*, viz. at *B*, &c. As will Evidently appear to any One who rightly considers, That

As a *Thread* just the *Length* of the *Diameter* of any *Circle*, having its *Two Ends* tyed together, and then *Moved* about a *Point* in the *Center* (viz. by making it a *double Radius*) will, by drawing another *Point* in its *Extremity*, Describe the *Periphery* of a *Circle* (vide *Definition* Page 280). Even so,

If a *Thread* just the *Length* of the *Transverse Diameter* (*TS*) have its *Two Ends* so fix'd upon the *Two Focus's* (*f* and *F*) as that it may be *Moved* about them, by drawing a *Point* in its *Extremity* (viz. at its full *Streach*) it will describe the true *Periphery* of an *Ellipsis*.

Now although this *Easie Way* of Describing, Or, as usually phrased, Drawing an *Ellipsis*, be *Mechanical*, and known even to most *Joyners*, *Carpenters*, &c. yet it gives as *Compleat* and *Clear* an *Idea* of that *Figure*, as any other way whatsoever; And by Describing it thus about its *Two Focus's*, as a *Circle* is about its *Center*, doth plainly shew those *Two Points* are not *Improperly* called *particular Centers* in *Definition 4. Sect. 3. Chap. 1.* For each of them bears much what the same respect to the *Ellipsis Periphery*, as the *Circle's Center* doth to its *Periphery*.

Sect. 4. To Describe or Delineat an Ellipsis several Ways.

There are several (*other*) *Ways* of Describing an *Ellipsis*, both *Geometrical* and *Numerically*, according to peculiar *Occasions*, but I shall only mention *Two* or *Three* of them, leaving the rest to the *Learner's Genius*. Now in order to that *Work*, it will be convenient to consider what *Lines* are *Requisite* to *limit* or bound its *Form*, which I take to be chiefly these following.

1. If the *Transverse* and *Conjugate* are given, the *Ellipsis* is perfectly Limited (vide *Confectary* Page 363.) For if *TS* and *Nn*, be set at *Right Angles* in their *Middle* at *C*, And *TC* Or *CS*, be set off from *N*, Or *n*, both *Ways* upon the *Transverse* to *f*, and *F*; (viz. make $fN = TC = NF$) Then will those *Points f* and *F*, be the *two Focus's* (Per 4th Step of the Last Process) And then the *Ellipsis* may be Described As above.

2. If

2. If the *Transverse Diameter* and *Latus Rectum* are given, the *Ellipsis* is truly *Limited*, because by them the *Conjugate* may be found, per Sect. 2.

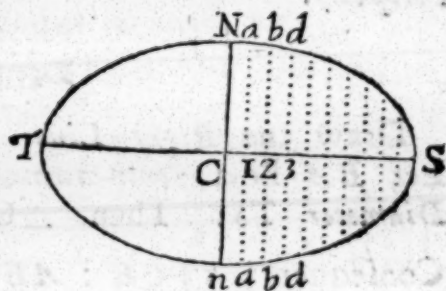
3. Or if only the *Transverse*, and the *Proportion* it hath, either to the *Conjugate* or *Latus Rectum*, be given, the *Ellipsis* is thereby *Limited*. As for Instance, suppose the given *Ratio* between the *Transverse* and *Conjugate* to be, As $a : To d$; Viz. $a : d :: TS : Nn$, then $\frac{TS \times d}{a} = Nn$, &c.

4. If either the *Transverse*, Or *Conjugate*, and the *Distance* of the *Focus* from the *Conjugate* be given, the *Ellipsis* is *Limited*, because by them the *Conjugate*, Or *Transverse* may be found.

These being premised, and the precedent Work a little considered, it must needs be *Easie* to *Describe* or *Delineate* any *Ellipsis* in *Plano*, either *Geometrically*; Or *Numerically*.

1. To *Describe* an *Ellipsis* Numerically by *Points*.

Suppose the *Transverse Diameter* $TS = 20$ and the *Conjugate* $Nn = 12$ (either *Inches* or any other *Equal Parts*) And let them cross each other at *Right Angles* in their *Middles*, As in the Point *C*.



Then will $TC = CS = 10$.

And $NC = Cn = 6$.

And it will $20 : 12 :: 12 : 7,2$ = the *Latus Rectum*.

Again $20 : 7,2$ Or rather take their *Ratio*.

Thus $\begin{cases} 1 : 0,36 :: 10 + 1 \times 10 - 1 : \square a.1. \\ 1 : 0,36 :: 10 + 2 \times 10 - 2 : \square b.2. \\ 1 : 0,36 :: 10 + 3 \times 10 - 3 : \square d.3. \end{cases}$ &c.

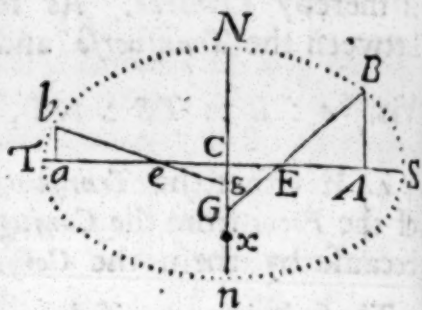
Viz. $\begin{cases} 100 - 1 \times 0,36 = \square a.1. \text{ Hence } \sqrt{99 \times 0,36} = 5,97 \&c. = a.1. \\ 100 - 4 \times 0,36 = \square b.2. \quad \sqrt{96 \times 0,36} = 5,88 \&c. = b.2. \\ 100 - 9 \times 0,36 = \square d.3. \quad \sqrt{91 \times 0,36} = 5,72 \&c. = d.1. \end{cases}$

If so many *Semi-ordinates* as may be thought convenient (the more the better) be found in this manner, and every one of them be set off at *Right Angles* from its respective Point in the *Transverse Diameter* each way; viz from 1 to *a*, from 2 to *b*, from 3 to *d*, &c. Then if a *Curve Line* be carefully drawn with an even *Hand*, through those *Extream Points* *a*, *b*, *d*, &c. it will be the *Ellipsis Periphery* required.

2. To

2. To Describe an Ellipsis Geometrically by Points.

Having the *Transverse* and *Conjugate Diameters* given, viz. *TS* and *Nn*, placed at *Right Angles* in their *Middles*, As before. Then from either End of the *Conjugate*, viz. *N* (or *n*) set off Half the *Transverse Diameter* to *x*. That is, make $Nx = TC$ (continuing the *Conjugate Nn*, when it is shorter than *TC*) Or, which is all one, make $Cx = TC - NC$. Then take any Point in the Line *Cx* at pleasure. Suppose it at *G*, and from that Point at *G*, set off the Distance *Cx*, to the *Transverse*, as at *E*, viz. make $GE = Cx$, and joyn the Points *GE* with a *Right Line*, produced so far beyond *E*, as to make $EB = NC$. Consequently $GB = TC$.



Then, I say, that wherever the Point *G* was taken between *C* and *x*, the Point *B* will just touch (or fall in) the *Ellipsis Periphery*.

Demonstration.

Draw the *Right Line BA* Perpendicular to *TS*. Viz. Let *BA* be a *Semi-ordinate* rightly Applied to the *Transverse Diameter TS*. Then $\triangle GCE$ and $\triangle BAE$ will be alike.

Consequen.	1	$CE : AE :: EG : EB$ Per Theorem 13.
1, And	2	$CE + AE : AE :: EG + EB : EB$ See Pag. 192
But	3	$CE + AE = CA$ $EG + EB = TC$ And $EB = NC$
Therefore	4	$CA : AE :: TC : NC$
6, in $\square$'s	5	$\square CA : \square AE :: \square TC : \square NC$
5, $\therefore$	6	$\frac{\square CA \times \square NC}{\square TC} = \square AE$
But	7	$\square NC - \square AB = \square AE$
That is,		$\square EB - \square AB = \square AE$
6, 7	8	$\frac{\square CA \times \square NC}{\square TC} = \square NC - \square AB$
8 $\times \square TC$	9	$\square CA \times \square NC = \square NC \times \square TC - \square AB \times \square TC$
9 $\pm$	10	$\square NC \times \square TC - \square CA \times \square NC = \square AB \times \square TC$
10, Analogy	11	$\square TC : \square NC :: \square TC - \square CA : \square AB$
That is,	12	$TC \times CS : \square NC :: TC + CA \times TC - CA : \square AB$

Which is according to the common Properties of the Ellipsis
Therefore the Point *B* is truly found.

Q. E. D.
Hen

Hence it follows, that if a convenient Number of such Lines as GEB , be so Drawn (as above directed) from the like Number of Points taken between C and x , &c. their Extream Points (As at B) will be those Points by which (with an even Hand) the Ellipsis may be truly Described. As before.

But if this be well understood, it will be very Easie to Conceive how to Describe an Ellipsis very readily, without drawing those Lines, by having a Thin straight narrow Ruler just the Length of TC , made somewhat Sharp at both Ends, upon which, from one of its Ends, set off the Length of NC . Then if the Point representing E , be gradually or easily Moved along the Transverse TS , and at the same Time, the Point or End representing G , be kept sliding close along the Conjugate Nn ; 'tis Evident from the Work above, that the End of the Ruler representing B , will, by that Motion, assign the true Periphery of the Ellipsis required. For by that Motion the Straight Edge of the Ruler doth supply an Infinite Number of the afore-said Lines; As will appear very Plain and Easie in Practice.

Scholium.

Now from hence was Deduced the first Invention of that well contrived Instrument for Drawing an Ellipsis by one Motion, commonly called the Elliptical Compasses, being usually made of Brass, and composed of Three Parts, Two of which represent (or rather supplie) the Transverse and Conjugate Diameters set together at Right Angles; And the Third Part is a Moveable Ruler, which performs the Office of the Last mentioned Thin Ruler. But because the making of it is so well known to most Mathematical Instrument Makers,

Especially to that Accurate and Ingenious Artist Mr. John Rowly, Mathematical Instrument Maker under St. Dunstan's Church in Fleetstreet, London; who for his great Skill in Conceiving, Framing, and Graduating all kind of Mathematical Instruments; may, I believe, be justly called one of the best Workmen of his Trade in Europe.

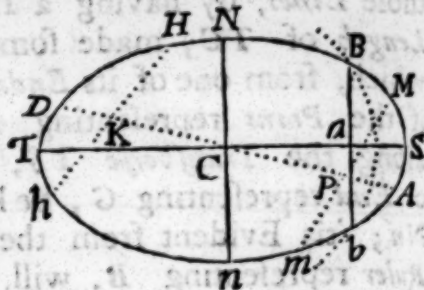
I think it needless therefore to give a particular Description of that Instrument.

Also from hence came that Ingenious Invention of making Engines for Turning all sorts of Elliptical Or Oval Work, as Oval Boxes, Picture Frames, &c.

Sect. 5. Any Ellipsis being given; To find its Transverse and Conjugate Diameters.

Suppose the given Ellipsis to be $TNSn$ (in the Annexed Scheme) in which let it be required to find the Transverse Diameter TS , and its Conjugate Nn .

Draw within the Ellipsis any Two Right Lines Parallel to each other, as Hh and Mm ; And Bisseth those Lines, viz. find the Middle Point of each, as at K and P . Then through those Points K and P , draw a Right Line, as DA , and it will be a Diameter; For it will Divide the Ellipsis into Two Equal Parts (See Defi. 1. Page 357.) Consequently the Middle of DA will be the true Middle or common Center of the Ellipsis, As at C .



For 'tis the Nature or Property of all Diameters, howsoever they are Drawn in any Ellipsis (As it is in a Circle) to cut or cross one another in the common Center, or Middle of the Figure, As at C .

Upon the Point C , Describe an Arch of any Circle that will cut the Ellipsis Periphery in Two Points, As at B and b ; then joyn those Points Bb , with a Right Line, and it will be an Ordinate, through whose Middle (As at a) and the common Center C , the Transverse Diameter TS must pass.

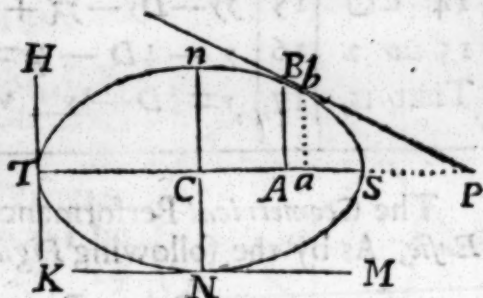
For $BS = Sb$, and Ba is at Right Angles with TS ; Therefore the Line Bb is an Ordinate Rightly Applied to TS the Transverse Diameter. And if through the Point C there be Drawn the Right Line Nn Parallel to Bb , it will become the Conjugate; As was required.

Sect. 6. To Draw a Tangent or Right Line that may touch the Ellipsis Periphery in any Assigned Point.

The Drawing of Tangents to, Or from any Assigned Point in the Ellipsis Periphery, admits of Three Cases.

Case 1. If it be required to Draw a Tangent that may Touch the Ellipsis in either of the Extreame Points of its Transverse Diameter, As at T or S , it is plain, the Tangent must be Drawn Parallel to the Conjugate Diameter Nn , As NK in the following Figure is supposed to be,

Case 2. Or if the Tangent must be Drawn to Touch the Ellipsis^s in either of the Extream Points of its Conjugate Diameter, As a^t N or n, 'tis as Evident that it must be Drawn Parallel to the Transverse Diameter TS, as KM. Consequently, if that Tangent, and the Transverse were both Infinitely Continued, they would never meet.



Case 3. But if it be required to Draw a Tangent that may Touch the Ellipsis in any other Point, as at B, &c.

Then if the Tangent and the Transverse Diameter be both Continued, they will meet in some Point, as at P; And those Two Points (viz. B and P) do so mutually depend upon each other, that one of them must be Assigned in Order to find the other, that so the Tangent may by them be truly Drawn.

Let $D = TS$. $y = AS$: and $z = AP$. Then if y be given, z may be found by this, Theorem $\left\{ \frac{Dy - yy}{\frac{1}{2}D - y} = z \right\}$.

Or if z be given, y may be found by this Theorem.

$$\text{Theorem } \left\{ \frac{D + z}{2} \pm \sqrt{\frac{DD + zz}{4}} = y \right\}$$

Demonstration.

Draw the Semi-Ordinate ba , as in the Figure; then will $\triangle BAP$ and $\triangle baP$ be alike. Put $x = Aa$ the Distance between the Two Semi-ordinates (viz. between BA and ba) which we suppose Infinitely small.

Then	1	$z : z - x :: BA : ba$ Per Theorem 13.
But	2	$D - y \times y : D - y + x \times y - x :: \square BA : \square ba$
That is,	3	$Dy - yy : Dy - yy + 2yx - Dx - xx :: \square BA : \square ba$
1 in $\square$'s	4	$zz : zz - 2zx + xx :: \square BA : \square ba$
Suppose	5	$x = 0$ That so x may be every where Rejected
3, Then	6	$Dy - yy : Dy - yy + 2y - D :: \square BA : \square ba$
4, And	7	$zz : zz - 2z :: \square BA : \square ba$
6, 7	8	$Dy - yy : Dy - yy + 2y - D :: zz : zz - 2z$
8 $\therefore$	9	$2yzz - Dzz = 2yyz - 2Dyz$
9 $\div 2z$	10	$yz - \frac{1}{2}Dz = yy - Dy$
10 $\div \frac{1}{2}$	11	$\frac{1}{2}Dz - yz = Dy - yy$

11 ÷	12	$z = \frac{Dy - yy}{\frac{1}{2}D - y}$	{ Which is the 1st Theorem; And gives the following Analogy.
Analogy	13	$\frac{1}{2}D - y : y :: D - y : z$	Viz. $CA : SA :: TA : AP$
10 — yz	14	$yy - Dy - yz = -\frac{1}{2}Dz$	
14 C□	15	$yy - Dy - yz + \frac{1}{4}DD - \frac{1}{4}Dz + \frac{1}{4}zz = \frac{1}{4}DD + \frac{1}{4}zz$	
15 uv 2	16	$y - \frac{1}{2}D - \frac{1}{2}y = \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$	
That is,	17	$y = \frac{1}{2}D + \frac{1}{2}y \pm \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$	{ Which is the 2d Theor. Q. E. D

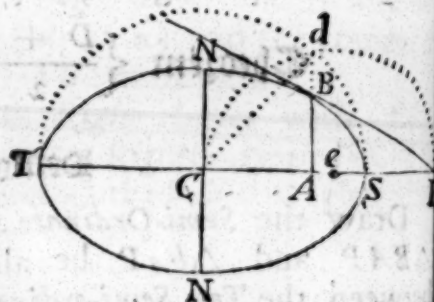
The Geometrical Performance of these Two Theorems is very Easie, As by the following Figure.

1. Suppose the Point *B* in the Ellipsis Periphery were given, and it were required to find the Point *P*, &c.

Make *TC* Radius, and upon the Common Center *C* Describe the Semi-circle *TdS*; and joyn the Points *C* and *d* with a Right Line; Then Bisect that Line (Per Prob. 2. Page 287.) and mark the Point where the Bisecting Line would cross the Transverse, as at *e*. Upon that Point *e*, with the Radius *Ce* (or *Cd*) Describe another Semi-circle, producing the Transverse Diameter to its Periphery, and it will assign the Point *P*.

For if $D = TS$. $y = AS$. and $z = AP$. As before.

Then	1	$D - y \times y = \square dA$
And	2	$\frac{1}{2}D - y \times z = \square dA$
For	3	$TA : dA :: dA : SA$
And	4	$CA : dA :: dA : AP$
But		$CA = \frac{1}{2}D - y$, &c.
1, 2	5	{ $\frac{1}{2}Dz - yz = Dy - yy$ As at the 11 Step before.



Therefore the Point *P* is truly found; Consequently, if a Right Line be drawn through those Points *B* and *P*, it will be the Tangent required; according to the First Theorem.

2. The Converse of this is as Easie; to wit, if the Point *P* be given, thence to find the Point *B* in the Ellipsis Periphery. Thus, Circumscribe Half the Ellipsis with the Semi-circle *TdS*, As before; And Bisect the Distance between the Points *C* and *P*, As at *e*; viz. Let $Ce = eP$. Then making *Ce* Radius, upon the Point *e* Describe the Semi-circle *CdP*. And from the Point where the Two Semi-circles Intersect or Cross each other, as at *d*, Draw the Right Line *dA* Perpendicular to the Transverse *TS*, and it will Assign the Point *B*.

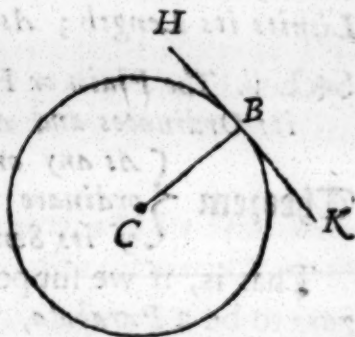
Contd.

Contact B in the Ellipsis Periphery, through which the Tangent must pass.

But a *Practical Method* of Drawing Tangents to any Assigned Point in the Ellipsis Periphery, may (without finding the *aforesaid Point P*) be easily deduced from the following Property of Tangents drawn to a Circle, which is this.

If to any Radius of a Circle, As CB, there be Drawn a Tangent Line (As HK) to Touch the Radius at the Point B, The Two Angles which the Tangent makes with the Radius, will always be Two Right Angles (16, 17, 18, 19 *Eūclid.* 3.)

That is, $\angle HBC = \angle CBK = 90^\circ$.



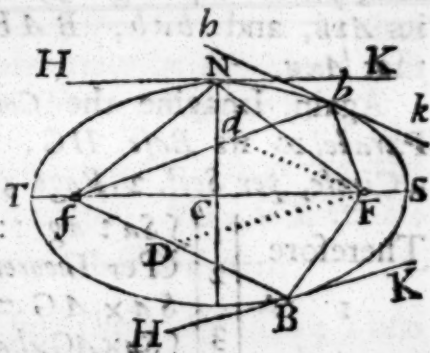
In like manner the Two Angles made between the Tangent and the Two Lines Drawn from the Focus's of an Ellipsis to the Point of Contact, will always be Equal; but not Right Angles, save only at the Two Ends of the Transverse Diameter.

These being well considered, and compared with what hath been said in Page 366 it must needs be *easy* to understand the following way of Drawing Tangents, to any Assigned Point in the Ellipsis Periphery; which is thus.

Having by the Transverse and Conjugate Diameters, found the Two Focus's *f* and *F*, per *Sec.* 3. From them Draw Two Right Lines to meet each other in the Assigned Point of Contact, as *fb* and *Fb* (Or *fB* and *FB*) in the Annexed Figure. Next set off (*viz.* make) $bd = bF$ (Or $BD = BF$) and joyn the Points *Fd* (Or *FD*) with a Right Line.

Then, I say, if a Right Line be Drawn through the Point of Contact *b* (or *B*) Parallel to *dF* (or *DF*) it will be the Tangent required.

For it is plain, that as the $\angle fNH = \angle FNK$ when the Tangent is Parallel to the Transverse Diameter; even so is the $\angle fbb = \angle Fbk$ (and $\angle fBH = \angle FBK$) And will be every where so, as the Point of Contact *b* (Or *B*) and its Tangent, is carried about the Ellipsis Periphery with the Lines *fbF* (Or *fBF*).



C H A P. III.

Concerning the Chief Properties of every Parabola.

Note, In every Parabola, the Intercepted Diameter, Or that Part of its Axis which is between the Vertex and that Ordinate which Limits its Length; As Sa Or SA , &c. is called *Abscissa*.

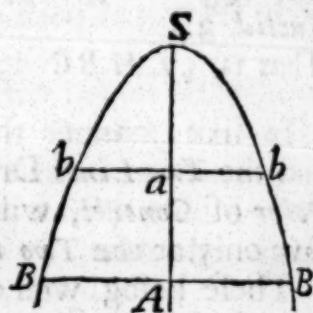
Sect. 1. The Plain or Figure of every Parabola, is Proportioned by its Ordinates and *Abscissa's*; as in the following Theorem.

Theorem { As any one *Abscissa* : Is to the Square of its Semi-ordinate :: So is any other *Abscissa* : To the Square of its Semi-ordinate.

That is, if we suppose the Annexed Figure to be a Parabola, wherein Sa , and SA , are *Abscissa's*, and bab , BAB Ordinates Rightly applied, it will

be $Sa : \square ba :: SA : \square BA$ } wheresoever
Or $Sa : SA :: \square ba : \square BA$ } the Points a ,
And so for any other *Abscissa*, &c. A , are taken

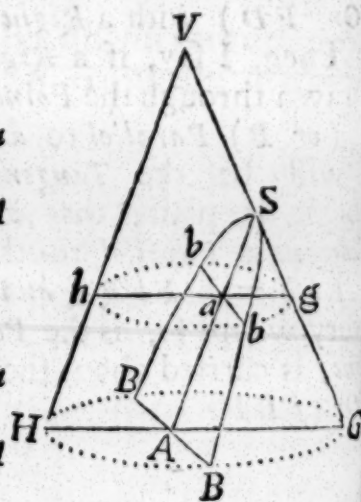
Demonstration.



Let the following Figure HVG represent a Right Cone, Cut into Two Parts by the Right Line SA , Parallel to its Side VH . Then the Plain of that Section, viz. $BbSbB$ will be a Parabola, per Sect. 4. Page 358. wherein let us suppose SA to be its Axis, and bab , BAB to be Ordinates Rightly applied to that Axis.

Again, Imagine the Cone to be Cut by the Right Line hg Parallel to its Base HG . Then will hg be the Diameter of a Circle, per Sect. 2. Page 357. And $\triangle Sag$ like to $\triangle SAG$.

Therefore	1	{ $Sa : ag :: SA : AG$
	2	{ Per Theorem 13.
1 $\therefore$		$Sa \times AG = SA \times ag$
2 $\times ba$	3	{ $Sa \times AG \times ba = SA \times ag \times ba$
		{ Per Axiom 3.
But	4	{ $HA = ba$ because SA
		{ is Parallel to VH
And {	5	{ $\square BA = AG \times HA$ } Per
		{ $\square ba = ag \times ba$ } Lemma
		{ P. 357.
3, 4, 5	6	{ $Sa \times \square BA = SA \times \square ba$
		{ Per Axiom 5.
6, Analogy	7	{ $Sa : \square ba :: SA : \square BA$
		{ Vide Page 194.



Q. E. D.

The

These Proportions being proved to be the Common Property of every Parabola; all that is farther required about that Section or Figure, may from thence be easily deduced.

Sect. 2. To find the Latus Rectum, Or Right Parameter of any Parabola.

The Latus Rectum of a Parabola, hath the same Ratio or Proportion to any Abscissa and its Semi-ordinate; As the Latus Rectum of an Ellipsis, hath to its Transverse and Conjugate Diameters. And may be found by this Theorem.

Theorem { As any Abscissa : Is in Proportion to its Semi-ordinate :: So is that Semi-ordinate : To the Latus Rectum.

Let L = the Latus Rectum.

Then	1	$Sa : ba :: ba : L$	{ where ever the Points a , and A , are taken in the Axis.
And	2	$SA : BA :: BA : L$	
1 ::	3	$\frac{\square ba}{Sa} = L$	Or $Sa \times L = \square ba$
2 ::	4	$\frac{\square BA}{SA} = L$	Or $SA \times L = \square BA$
3 = 4	5	$\frac{\square BA}{SA} = \frac{\square ba}{Sa}$	Per Axiom 5.
5 x	6	$Sa \times \square BA = SA \times \square ba$ Which gives this Analogy.	
Analogy	7	$Sa : \square ba :: SA : \square BA$ The same as at the 7th Step of the last Process; Therefore L (thus found) is the true Latus Rectum by which all the Ordinates may be Regulated and found, according to its Definition in Section 4. Page 358.	

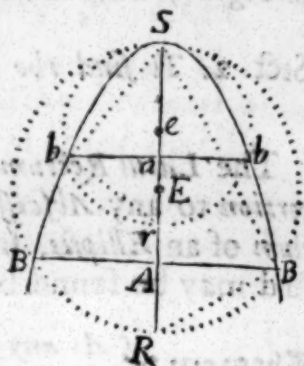
For by the Third Step $Sa \times L = \square ba$. And by the 4th Step $SA \times L = \square BA$. Consequently $\sqrt{Sa \times L} = ba$ And $\sqrt{SA \times L} = BA$; and so for any other Ordinate.

Or if the Ordinates are given, To find their Abscissa's; Then it will be, $L : ba :: ba : Sa$. And $L : BA :: BA : SA$, &c.

Consequently $\frac{\square ba}{L} = Sa$. And $\frac{\square BA}{L} = SA$, &c.

From the Consideration of these Proportions, it will be easy to conceive how to find the Latus Rectum Geometrically. Thus,

Joyn the Vertical Point S of the Axis, and either Extream Point of any Ordinate, as B , (Or b) with a Right Line, viz. SB (or Sb) and Bissect that Line (per Problem 2. Page 287.) marking the Point where the bissecting Line doth Intersect or Cross the Axis, as at E (Or e) and with the Radius SE (or Se) upon the Point E (Or e) Describe a Circle, (As in the Annexed Figure) Then will the Distance between the Ordinate, and that Point where the Circle's Periphery cuts the Axis viz. AR (Or ar) be the true Latus Rectum required.



For $SA:BA::BA:AR$. And $Sa:ba::ba:ar$. Per Theor. 13. Therefore $AR=L$. And $ar=L$ by the 1st and 2d Steps above.

Confectary.

From these Proportions of finding the Latus Rectum, it will be easie to deduce, and Demonstrate this following Theorem.

Theorem { As the Latus Rectum : Is to the Sum of any Two Semi-ordinates :: So is the Difference of those Two Semi-ordinates : To the Difference of their Abscissas.

Suppose any Right Line drawn within the Parabola, as bD , Parallel to its Axis SA , Then will that Line (viz. bD) be a Diameter (per Defi. 5. Page 359) which will make $ED=AB+ab$, $DB=AB-ab$, And $bD=SA-Sa$. Then it will be $L:ED::DB:bD$. According to the Theorem.

Demonstration.

First	1	$\left\{ \begin{array}{l} SA = \frac{\square BA}{L} \text{ Per Step 2.} \\ \text{of the Last Proceß.} \end{array} \right.$	
And	2	$\left\{ \begin{array}{l} Sa = \frac{\square ba}{L} \text{ Per Step 1.} \\ \text{of the Last Proceß.} \end{array} \right.$	
1 — 2	3	$SA - Sa = \frac{\square BA - \square ba}{L}$	
3 × L	4	$SA - Sa \times L = \square BA - \square ba$	
But	5	$\square BA - \square ba = BA + ba \times BA - ba$	Which gives the following Analogy.
4, = 5,	6	$SA - Sa \times L = BA + ba \times BA - ba$	
6, Analogy	7	$L : BA + ba :: BA - ba : SA - Sa$	
Or	8	$L : ED :: DB : bD$	

This peculiar Property of the Parabola, was first Published Anno 1684. by one Mr. Thomas Baker, Rector of Bishop Nymton in Devonshire, in a Treatise Intitul'd, The Geometrical Key; Or, The Gate of Equations Unlocked; wherein he hath shewed the Geometrical Construction and Solution of all Cubick, and Biquadratick Affected Equations by one general Method, which he calls a Central Rule; deduced from this peculiar Property of the Parabola.

Sect. 3. To find the Focus of any Parabola.

The Focus of every Parabola, is that Point in its Axis through which the Latus Rectum doth pass. (vide Definition 3. Sect. 4. Page 359) Therefore its Distance from the Vertex of the Parabola may be easily found, either by the Latus Rectum it self, or by any other Ordinate and its Abscissa.

Thus; Suppose the Point at F to be the Focus; S the Vertex; the Ordinate RFR = L the Latus Rectum; And b a b any other Ordinate.

Then will $SF = \frac{1}{4}L$. Or $SF = \frac{\square ba}{4Sa}$



Demonstration.

First	1	$SF \times L = \square FR$	Per Sect. 2. Page 375
And	2	$FR = \frac{1}{2}L$	For the Ordinate RFR = L as above.
2 $\odot$ 2	3	$\square FR = \frac{1}{4}\square L = \frac{1}{2}L \times \frac{1}{2}L$	
1, = 3,	4	$SF \times L = \frac{1}{4}\square L$	
4 $\div$ L	5	$SF = \frac{1}{4}L$	As per Definition 4. Sect. 4. Page 359
Again	6	$\frac{\square ba}{Sa} = L$	by the Third Step in Page 375.
Consequ.	7	$\frac{\square ba}{4Sa} = \frac{1}{4}L$	&c. As above. Q. E. D.

Sect. 4. To Describe or Draw a Parabola several Ways.

Note, There are Two or Three Ways of Drawing a Parabola Instrumentally at one Motion; but because those Instruments or Machines are not only too perplext for a Learner to manage, but also a little Subject to Errour, I have therefore chosen to shew how that Figure may be (the best) drawn by a convenient Number of Points, viz. Ordinates found, either Numerically, Or Geometrically, according to the Data; which, if the Work of the Three last Sections be well considered, must needs be very Easie,

1. If any *Ordinate* and its *Abscissa* are given, there may by them be found as many *Ordinates* as you please to assign or take *Points* in the *Parabola's Axis*, (per *Seçt. 1. Page 374.*) and the *Curve* of the *Parabola* may be *Drawn* by the *Extream Points* of those *Ordinates*, as the *Ellipsis* was, *Page 367.*

2. If the *Latus Rectum*, and either any *Ordinate*, or its *Abscissa* are given, then any assigned *Number* of *Ordinates* may by them be found (per *Seçt. 2. Page 375.*) either *Numerically*, or *Geometrically*. &c.

3. If only the *Distance* of the *Focus* from the *Vertex* of the *Parabola* be given; any assigned *Number* of *Ordinates* may be found by it. For $SF = \frac{1}{4}L$ the *Latus Rectum*, And $\frac{1}{2}L = FR$ as in the *Last Section*; And it will be, As SF : Is to $\square FR$:: So is any other *Abscissa* (viz. SA , Or SA , &c.) : To the *Square* of its *Semi-ordinate* (viz. $\square ba$, Or $\square BA$) according to the common *Property* of the *Parabola*.

Altho' any of these *Ways* of finding the *Ordinates* are easie enough; yet that *Way* which may be deduced from the following *Proposition*, will be found much more easie and ready in *Practice*.

Proposition { The Sum of any *Abscissa* and *Focal Distance* from the *Vertex*, will be Equal to the *Distance* from the *Focus* to the *Extream Point* of the *Ordinate* which cuts off that *Abscissa*.

For Instance, Suppose S to be the *Vertex* of any *Parabola*; the *Point* F to be its *Focus*; and AB any *Semi-ordinate* *Rightly Applied* to its *Axis* SA . Then, I say, where ever the *Point* A is taken in the *Axis*, it will be $SA + SF = FB$. Consequently, if $Sf = SF$ it will be $fA = FB$.



Demonstration.

First	1	$SF = \frac{1}{4}L$	by the 7th Step. <i>Seçt. 3.</i>
Ergo	2	$fA = FA + \frac{1}{2}L$	by <i>Construction</i> above.
2 @ 2	3	$\square fA = \square FA + FA \times L : + \frac{1}{4}LL$	
Again	4	$SA = FA + \frac{1}{4}L$	by the <i>Supposition</i> and <i>Figure</i> .
4 x L	5	$SA \times L = FA \times L : + \frac{1}{4}LL$	But $SA \times L = \square AB$.
Ergo	6	$\square AB = FA \times L : + \frac{1}{4}LL$	
3 - 6	7	$\square fA - \square AB = \square FA$	Conse. $\square fA = \square FA + \square AB$
But	8	$\square FA + \square AB = \square FB$	Per <i>Theorem 11.</i>
Ergo	9	$\square fA = \square FB$	
9 uw 2	10	$fA = FB$	Q. E. D.

This

This Proposition being well understood, it will be very easily applied to Practice; supposing the Focal Distance given, or any other Data by which it may be found.

Thus, Draw any Right-line to represent the Parabola's Axis, and from its Vertical Point as at *S*, set off the Focal Distance both upwards and downwards, viz. make $Sf = SF$ the Distance of the given Focus from the Vertex: As in the last Scheme. Then by the Proposition it is evident; that if never so many Lines be drawn Ordinately at Right-angles to the Axis, the true Distance between the Point *f* out of the Parabola, and any of those Lines (or Ordinates) being Measur'd or set off from the Focus *F* to the same Line, (or Ordinate) it will Assign the true Point in that Line through which the Curve must pass. That is, it will shew the true Limits or Length of that Ordinate. As at *B* in the last Scheme.

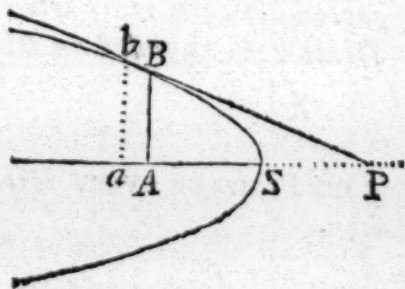
Proceeding on in the very same manner, from Ordinate to Ordinate, you may with great Expedition and Exactness, find as many Ordinates (or rather their Points only, like *B*) as may be thought convenient, which being all joyn'd together with an even Hand, will form the Parabola required.

N. B. The more Ordinates (or their Points) there are found, and the nearer they are to one another, the easier and exacter may the Curve of the Parabola be drawn. The same is to be observed when any other Curve is required to be drawn by Points.

Sect. 5. To draw a Tangent to any given Point in the Curve of a Parabola.

Tangents are very easily drawn to the Curve of any Parabola.

For, Supposing *S* to be its Vertex, *B* the Point of Contact; viz. the Point where the Tangent must touch the Curve. And *P* the Point where the Tangent will intersect (or meet with) the Parabola's Axis produced. Then if from the Point of Contact *B*, there be drawn the Semi-ordinate *BA* at Right-angles to the Axis *SA*, where so ever the Point *A* falls in the Axis, it will be $SP = SA$.



Demonstration.

Draw the Semi-ordinate *ba* (as in the Figure) then will

C c c 2

the

the $\triangle BAP$, and $\triangle b a P$ be alike. Let $y = AS$ the *Ab-scissa*, and $z = SP$, put $x = Aa$ the Distance between the Two *Semi-ordinates*; which we suppose to be *Infinitely near* each other, as in the *Ellipsis* page 371.

Then	1	$y + z : BA :: y + z + x : ba$	Per Theorem 13.
1, Or	2	$y + z : y + z + x :: BA : ba$	Vide Page 192.
Again	3	$y : \square BA :: y + x : \square ba$	Per Theorem Page 374.
3, Or	4	$y : y + x :: \square BA : \square ba$	
2 in $\square$'s	5	$yy + 2yz + zz : yy + 2yz + 2yx + zz + 2zx + xx :: \square BA : \square ba$	
4, 5	6	$y : y + x :: yy + 2yz + zz : yy + 2yz + 2yx + zz + 2zx + xx$	
6 ::	7	$\begin{cases} yy + 2yz + yx + zz + 2zx + \frac{zzx}{y} = \\ yy + 2yz + 2yx + zz + 2zx + xx \end{cases}$	
That is,	8	$\frac{zzx}{y} = yx + xx$	Consequently $\frac{zz}{y} = y + x$
Suppose	9	$x = 0$	And Rejected, as in the <i>Ellipsis</i> Page 371.
Then	10	$\frac{zz}{y} = y$	Consequently $zz = yy$
Now 2	11	$z + y$	That is, $SP = SA$

Q. E. D.

CHAP. IV.

Concerning the chief Properties of the Hyperbola.

Note, Any part of the *Axis* of an Hyperbola, which is intercepted between its *Vertex* and any *Ordinate*, (viz. any intercepted *Diameter*) is called an *Ab-scissa*; as in the *Parabola*.

Sect. I.

The Plain of every Hyperbola, is proportioned by this General Theorem.

Theorem. $\left\{ \begin{array}{l} \text{As the Sum of the Tranverse and any Ab-scissa} \\ \text{Multiplied into that Ab-scissa: Is to the Square of} \\ \text{its Semi-ordinate} :: \text{So is the Sum of the Transverse} \\ \text{and any other Ab-scissa Multiplied into that Ab-scissa} \\ \text{To the Square of its Semi-ordinate.} \end{array} \right.$

That

That is, if TS , be the Transverse Diameter;

And $\begin{cases} Sa, SA, \text{ Abscissa's.} \\ ba, BA, \text{ Semi-ordinates.} \end{cases}$

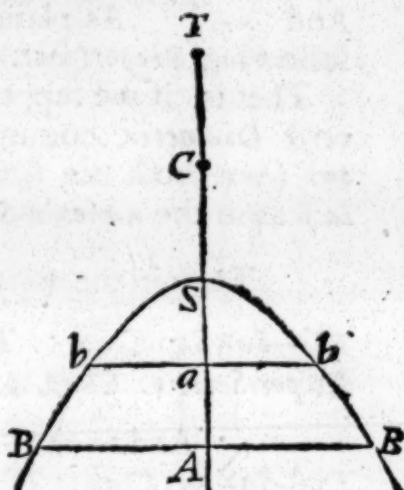
Then is $\begin{cases} Ta = TS - Sa. \\ TA = TS + SA. \end{cases}$

And it will be

$$Ta \times Sa : \square ba :: TA \times SA : \square BA:$$

That is,

$$TS - Sa \times Sa : \square ba :: TS + SA \times SA : \square BA. \\ \&c.$$

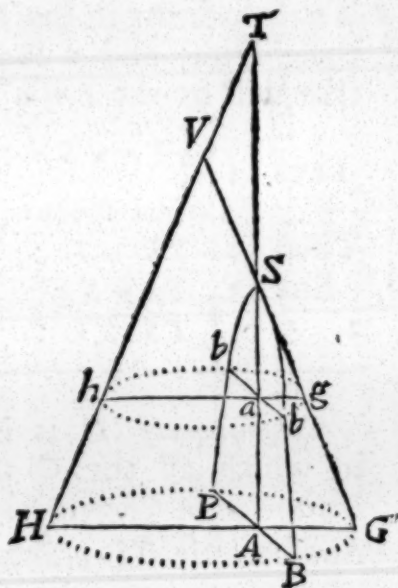


Demonstration.

Let the following Figure HVG , represent a Right Cone cut into Two Parts by the Right-line SA . Then will the Plain of that Section be an Hyperbola (per Sect. 5. Chap. 1.) in which let SA be its Axis or intercepted Diameter, bab , and BAB , Ordinates Rightly Apply'd, (as before in the Parabola) and TS its Transverse Diameter.

Again, if the Cone is supposed to be Cut by hg , Parallel to its Base HG , it will also be the Diameter of a Circle, &c. As in the Ellipsis and Parabola.

Then will the $\triangle Sga$ and $\triangle SGA$, be alike, also the $\triangle Tab$ and $\triangle TAH$ will be alike, therefore it



will be	1	$Sa : ag :: SA : AG$
And	2	$Ta : ab :: TA : AH$
1 ::	3	$Sa \times AG = SA \times ag$
2 ::	4	$Ta \times AH = TA \times ab$
3 x 4	5	$\begin{cases} Sa \times Ta \times AG \times AH = \\ = SA \times TA \times ag \times ab \end{cases}$
But	6	$ag \times ab = \square ab$
And	7	$\begin{cases} AG \times AH = \square AB \\ \text{Per Lemma Page 357.} \end{cases}$
3, 6, 7	8	$\begin{cases} Sa \times Ta \times \square AB = \\ = SA \times TA \times \square ab \end{cases}$
8, Anal.	9	$Sa \times Ta : \square ab :: SA \times TA : \square AB, \&c.$

Q. E. D.
These

These Proportions are the common Property of every Hyperbola, and do only differ from those of the Ellipsis in the Signs + And —. As plainly appears in all the following Proportions.

That is, if we suppose TS, the Transverse Diameter, common to both Sections (viz. both the Ellipsis and Hyperbola) as in the annex'd Scheme.

Then in the Ellipsis it will be

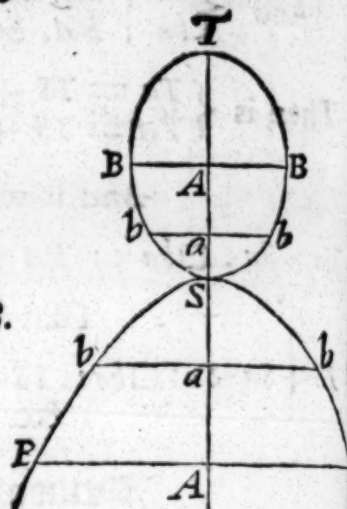
$$TS - Sa \times Sa : \square ab :: TS - SA \times SA : \square AB.$$

As per Sect. 1. Chap. 2.)

And in the Hyperbola it is

$$TS + Sa \times Sa : \square ab :: TS + SA \times SA : \square AB.$$

As above.



And therefore all that is further required in the Hyperbola, may (in a manner) be found as in the Ellipsis, due regard being had to Changing of the Signs.

Sect. 2. To find the Latus Rectum, Or Right Parameter of any Hyperbola.

From the last Proportion, take either of the Antecedents and its Consequent, viz. either $Ta \times Sa : \square ab$. Or, $TA \times SA : \square AB$. To them bring in the Transverse TS for a Third Term, and by those Three find a Fourth Proportional, (As in the Ellipsis) and that will be the Latus Rectum.

Thus	1	{	$Ta \times Sa : \square ab :: TS : \frac{\square ab \times TS}{Ta \times Sa} = \text{the Lat}$
Then	2		$TS : L :: Ta \times Sa : \square ab$
But	3		$Ta \times Sa : \square ab :: TA \times SA : \square AB.$
2, 3	4		$TS : L :: TA \times SA : \square AB, \&c.$

tus Rectum, which call L (As in the Parabola.) Therefore

Consequently L is the true Latus Rectum, or Right Parameter by which all the Ordinates may be found. According to its Definition in Chapter 1.

And because $TS + Sa = Ta$. Let it be $TS + Sa$, instead of Ta

$$\text{Then it will be } \frac{\square ab \times TS}{TS \times Sa : + \square Sa} = L.$$

And in the Ellipsis it would be,
See Sect. 2. Chap. 2.

$$\frac{\square ab \times TS}{TS \times Sa : - \square Sa} = LR = L$$

Sect. 3. To find the Focus of any Hyperbola.

The Focus being that Point in the Hyperbola's Axis, through which the Latus Rectum must pass (as in the Ellipsis and Parabola) it may be found by this Theorem.

Theorem { To the Rectangle made of Half the Transverse into Half the Latus Rectum; Add the Square of Half the Transverse; the Square Root of that Sum, will be the Distance of the Focus from the Center of the Hyperbola.

Demonstration.

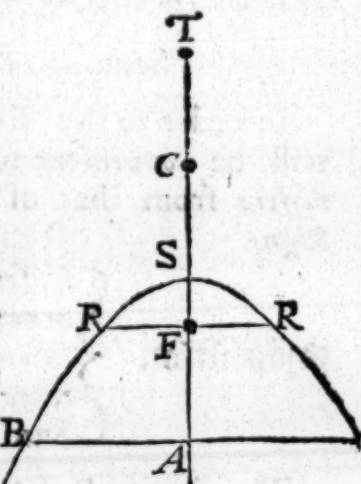
Suppose the Point at F, in the Annexed Scheme, to be the Focus sought. Then will $FR = \frac{1}{2}L$.

Let $TC = CS$ be Half the Transverse. Then is the Point C called the Center of the Hyperbola (for a Reason that shall be hereafter shewed.)

Again, Let $CS = d$. And $SF = a$.

Then
That is,
1
2
3
4
5
6
Or, 5, -d

1	$2d : L :: 2d + a \times a : \frac{1}{2}LL$
2	$TS : L :: TS + SF \times FS : \square FR$
3	$\frac{1}{2}dL = 2da + aa$
4	$dd + \frac{1}{2}dL = dd + 2da + aa$
5	$\sqrt{dd + \frac{1}{2}dL} = d + a = FC$
6	$\sqrt{dd + \frac{1}{2}dL} - d = a = SF$



In the Ellipsis it is, $2d : L :: 2d - a \times a : \frac{1}{2}LL$.
That is, $\frac{1}{2}dL = 2da - aa$, &c.

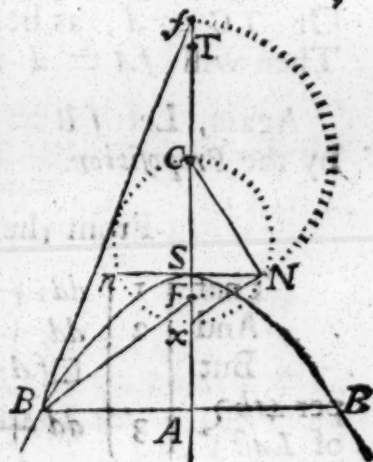
The Geometrical Effect of the Last Theorem is very easily performed, Thus.

Make $Sx = \frac{1}{2}L$, viz. Half the Latus Rectum; And let $CS = d$, as above.

Upon Cx (as a Diameter) describe a Circle; and at S the Vertex of the Hyperbola. Draw the Right Line nSN at Right-angles to Cx , Then joyn the Points C, N, with a Right Line, and it will be $CN = d + a = FC$.

For
That is,
2
3
But
3, 4
5
6

1	$CS : SN :: SN : Sx$ Per Fig.
2	$d : SN :: SN : \frac{1}{2}L$
3	$\frac{1}{2}dL = \square SN$
4	$dd + \square SN = \square CN$
5	$dd + \frac{1}{2}dL = \square CN$
6	$\sqrt{dd + \frac{1}{2}dL} = CN = d + a$, &c.



Now

Now here is not only found the *Distance* of the *Hyperbola's Focus*, either from its *Center C*, or *Vertex S*, But here is also found that *Right Line* usually called its *Conjugate Diameter*; viz. the *Line nSN*, which bears the same *Proportion* to the *Transverse* and *Latus Rectum* of the *Hyperbola*, as the *Conjugate Diameter* of the *Ellipsis* doth to its *Transverse* and *Latus Rectum*.

For in the *Ellipsis* $TS : Nn :: Nn : LR$. per *Sect. 2. Page 363*.

Consequently $\frac{1}{2}TS : \frac{1}{2}Nn :: \frac{1}{2}Nn : \frac{1}{2}LR$.

But $\frac{1}{2}TS = d$. $\frac{1}{2}Nn = SN$. And $\frac{1}{2}LR = \frac{1}{2}L$.
Therefore $d : SN :: SN : \frac{1}{2}L$. As at the *Second Step* above.

What Use the aforefaid *Line nSN* is of, in relation to the *Hyperbola*, will appear further on.

Sect. 4. To Describe an Hyperbola in Plano.

In order to the *Easie Describing* of an *Hyperbola* in *Plano*, it will be convenient to premise the following *Proposition*, which differs from that of the *Ellipsis* in *Sect. 3. Chap: 2.* only in the *Signs*.

Proposition { If from the *Focus's* of any *Hyperbola*, there be drawn Two *Right Lines*, so as to meet each other in any *Point* of the *Hyperbola's Curve*; the *Difference* of those *Lines* (in the *Ellipsis* it is their *Sum*) will be *Equal* to the *Transverse Diameter*.

That is, if *F* be the *Focus*, and it be made $Cf = CF$ (as in the last *Scheme*) then the *Point f* is said to be a *Focus* out of the *Section* (or rather of the *Opposite Section*) and it will be, $fB - FB = TS$.

Demonstration.

Suppose fC , Or $CF = z$, and $SA = x$. Let CS Or $TC = d$, as before.

Then will $fA = d + x + z$. And $FA = d + x - z$.

Again, Let $fB = b$, and $FB = b$. Then $2d = b - b$, by the *Proposition*.

From these *Substituted Letters* it follows,

That	1	$dd + 2dx + 2dz + xx + 2zx + zz = \square fA$
And	2	$dd + 2dx - 2dz + xx - 2zx + zz = \square FA$
But		$\square fA + \square AB = \square fB$. And $\square FA + \square AB = \square FB$
per 4 th of Last }	3	$dd + \frac{1}{2}dL = dd + 2da + aa = \square FC = zz$

3 — dd	4	$zz - dd = \frac{1}{2}dL$	
4 $\div \frac{1}{2}d$	5	$\frac{zz - dd}{\frac{1}{2}d} = L$	
Again	6	$2d : L :: 2d + xxx : \square AB$ Per com. Properties.	
5, 6	7	$2d : \frac{zz - dd}{\frac{1}{2}d} :: 2dx + xx : \square AB$	
7 ::	8	$\frac{2dzzx + zzzx - 2dddx - ddx}{dd} = \square AB$	
1 + 8	9	$\left\{ \begin{array}{l} dd + 2dx + 2dz + xx + 2zx + zz + \\ 2dzzx + zzzx - 2d^3x - ddx \end{array} \right. = \square fA + \square AB = hb$	
2 + 8	10	$\left\{ \begin{array}{l} dd + 2dx - 2dz + xx - 2zx + zz + \\ 2dzzx + zzzx - 2d^3x - ddx \end{array} \right. = \square fA + \square AB = bb$	
9 x dd	11	$d^4 + 2d^3z + 2ddzx + ddzz + 2dzzx + zzzx = ddbb$	
10 x dd	12	$d^4 - 2d^3z - 2ddzx + ddzz + 2dzzx + zzzx = ddbb$	
11 uw 2	13	$dd + dz + zx = db$	
12 uw 2	14	$dd - dz - zx = db$	
13 $\div d$	15	$d + z + \frac{zx}{d} = b$	Altho' the Equation at the 16th Step be in it self impossible, because z is greater than d, (per 4th Step) yet from thence it will be easie to conclude, that the Difference between d and $z + \frac{zx}{d}$ will give the true Value of b. As in the 17th Step.
14 $\div d$	16	$d - z - \frac{zx}{d} = b$	
16, Or	17	$z + \frac{zx}{d} - d = b$	
15 — 17	18	$2d = b - b$	

But because I would leave no Room for the *Learner* to doubt about changing the Equation, $d - z - \frac{zx}{d} = b$ into that of $z + \frac{zx}{d} - d = b$, it may be convenient to Illustrate the whole Process in Numbers, whereby, (I presume) it will plainly appear that $b - b = TS$.

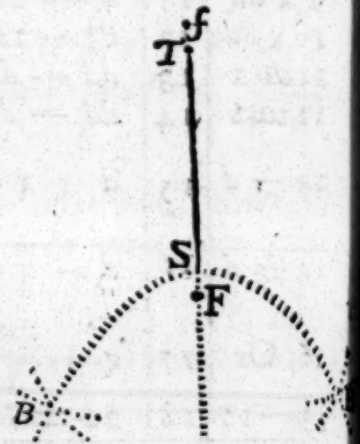
In order to that, Let the Transverse $TS = 2d = 12$. Then $d = 6$. Suppose the Abscissa $SA = x = 4$. And the Semi-ordinate $AB = 3$.

First	1	$TS + SA \times SA : \square AB :: TS : L$ Per Sect. 2.
1, Viz.	2	$12 + 4 \times 4 = 64 : 9 :: 12 : 1,6875 = L$
Again	3	$\sqrt{dd + \frac{1}{2}dL} = d + a = CF$ Per Sect. 3.
3, Viz.	4	$\sqrt{36 + 5,0625} = 6,408 = CF = z$
Then	5	$d + x + z = 6 + 4 + 6,408 = 16,408 = fA$
And	6	$d + x - z = 6 + 4 - 6,408 = 3,592 = fA$

5	⊙ 2	7	269,2224 = $\square fA$
6	⊙ 2	8	12,9024 = $\square FA$
	But	9	9 = $\square AB$ For $AB = 3$ by <i>Supposition</i> .
7	+ 9	10	278,2224 = $\square fA + \square AB = \square fB$
8	+ 9	11	21,9024 = $\square FA + \square AB = \square FB$
10	uv 2	12	16,68 = fB
11	uv 2	13	4,68 = FB
12 - 13		14	12,00 = $fB - FB = TS$. Which was to be proved

If this *Proposition* be truly understood, it must needs be *Easie* to conceive how to Describe the Curve of any *Hyperbola* very readily by *Points*, when the *Transverse Diameter*, and the *Focus* are given (Or any other Data by which they may be found, as in the precedent Rules.) Thus,

Draw any *streight Line* at Pleasure, in it set off the Length of the given *Transverse TS*, and from its Extream Points or Limits, viz. *T*, *S*, set off $Tf = SF$, the Distance of the given *Focus* (viz. the Point *f* without, and *F* within the Section, as before) that done, upon the Point *f* (as a Center) with any assumed *Radius* Greater than *TS*, describe an Arch of a Circle; then from that *Radius* take the *Transverse TS*, making their Difference a second *Radius*, with which upon the Point *F*, within the Section, describe another Arch to Cut or Cross the first Arch, as at *B*. Then will that Point *B*, be in the Curve of the *Hyperbola*, by the Last *Proposition*. And therefore its plain, that proceeding on in this manner, you may find as many Points (like *B*) as may be thought convenient (the more there are, and nearer they are together, the better) which being all joyned together with an even Hand (as in the *Parabola*) will form the *Hyperbola* required.



There are several other Ways of Delineating an *Hyperbola* in *Plano*; One Way is by finding a competent Number of *Ordinates*, as per *Sect. 1. &c.* But I think none so easie and expeditious as this *Mechanical Way*: I shall therefore for brevities sake pass over the rest, and leave them to the *Learners Practice*, as being easily deduced from what hath been already said.

Sect. 5. To Draw a Tangent to any given Point in the Curve of an Hyperbola.

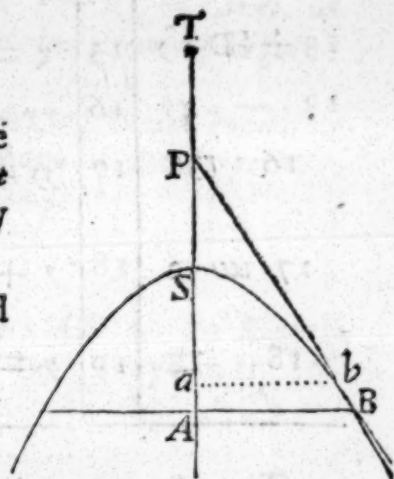
The Drawing of a Tangent that will touch any given Point in the Curve of an Hyperbola, may be easily performed by help of a Theorem; as in the Ellipsis, Sect. 6. Chap. 2.

Let $\begin{cases} D = TS & \text{the Transverse Diameter,} \\ L = & \text{the Latus Rectum.} \\ y = SA & \text{the Abscissa.} \end{cases}$

And $z = AP$ $\begin{cases} \text{the Distance between the} \\ \text{Ordinate, and that Point} \\ \text{in the Transverse cut by} \\ \text{the Tangent.} \end{cases}$

Then if y be given; z may be found by this Theorem $\left\{ \frac{Dy + yy}{\frac{1}{2}D + y} = z. \right.$

Which differs from that in the Ellipsis only in the Signs, vide Page 371.



Or if z be given; then y may be found by this Theorem

$$\text{Theorem } \left\{ \sqrt{\frac{DD + zz}{4}} : + \frac{1}{2}z - \frac{1}{2}D = y \right.$$

Demonstration.

Draw the Semi-ordinate ba , as in the Figure, and

put $x = Aa$ $\begin{cases} \text{an Infinite small Space between the Two Semi-} \\ \text{ordinates. As before in the Ellipsis, \&c.} \end{cases}$

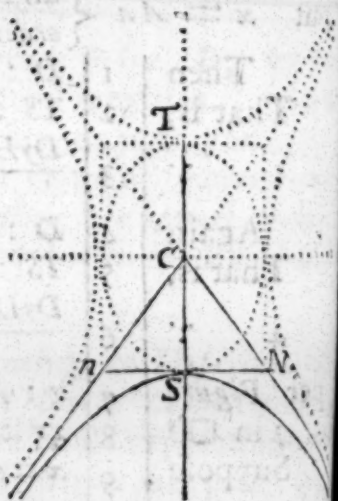
Then	1	$D : L :: Dy + yy : \square AB$
That is,	2	$TS : L :: TS + SA \times SA : \square AB$
1	3	$\frac{DyL + yyL}{D} = \square AB$
Again	4	$D : L :: Dy + yy - 2yx - Dx - xx : \square ab$
That is,	5	$TS : L :: TS + Sa \times Sa : \square ab$
4	6	$\frac{DyL + yyL - 2yxL - DxL - xxL}{D} = \square ab$
per Figure	7	$z : AB :: z - x : ab. \text{ Viz. } PA : AB :: Pa : ab$
7 in $\square$'s	8	$zz : \square AB :: zz - 2zx + xx : \square ab$
Suppose	9	$x = 0 \text{ and every where Rejected, as in the Ellipsis.}$
Then 3, 9	10	$zz : \frac{DyL + yyL}{D} :: zz - 2z : \square ab$
10	11	$\frac{DyLzz + yyLzz - 2DyLz - 2yyLz}{Dzz} = \square ab$

		12	$\left\{ \begin{array}{l} \frac{DyL + yyL - 2yL - DL}{D} = \\ DyLzz + yyLzz - 2DyLz - 2yyLz \end{array} \right.$
6, 11			$\frac{Dzz}{D}$
12 Reduc'd	13		$\frac{1}{2}Dz + yz = Dy + yy$
13 Analogy	14		$\frac{1}{2}D + y : y :: D + y : z. \text{ Viz. } CA : SA :: TA : AP$
13 $\div \frac{1}{2}D + y$	15		$z = \frac{Dy + yy}{\frac{1}{2}D + y} \text{ Which is the First Theorem:}$
13 $- zy$	16		$yy + Dy - zy = \frac{1}{2}Dz$
16 $\square C$	17		$yy + Dy - zy + \frac{DD - 2Dz + zz}{4} = \frac{DD + zz}{4}$
17 $uw 2$	18		$y + \frac{1}{2}D - \frac{1}{2}z = \sqrt{\frac{DD + zz}{4}}$
18 $\pm$	19		$y = \sqrt{\frac{DD + zz}{4}} : + \frac{1}{2}z - \frac{1}{2}D \left\{ \begin{array}{l} \text{Which is the} \\ \text{second Theorem.} \end{array} \right. \text{ Q. E. D.}$

The Geometrical Effecti^{on} of the First of these Theorems is very Easie; for by the 14th Step it is Evident, that there is Three Lines given to find a Fourth Proportional Line. Per Problem 3. Page 308.

Scholium.

From the Comparisons which have been all along made in this Chapter, between the Hyperbola and the Ellipsis, it will be Easie (even for a Learner) to perceive the Coherence that is in (or between) those Two Figures. - But for the better Understanding of what is meant by the Center and Assymptotes of an Hyperbola consider the annexed Scheme, wherein it is evident (even by Inspection) that the opposite Hyperbola's will always be alike, because they will always have the same Transverse Diameter common to both, &c. See Sect. 1. of this Chap. Also, that the middle Point, or common Center of the Ellipsis, is the common Center to all the four Conjugal Hyperbola's.



And the Two Diagonals of the Right-angled Parallelogram which Circumscribes the Ellipsis (or is Inscribed to the four Hyperbola's) being continued, will be such Assymptotes to those Hyperbola's, as are Defined, Chap. 1. Sect. 5. Defi. 4.

Sect. 6. To Draw the Asymptotes of an Hyperbola, &c.

Having found the *Latus Rectum* (per Sect. 2.) And the *Conjugate Diameter* nSN , in its true Position, per Sect. 3. Then through the Center C of the Hyperbola, and the Extream Points n , N , of its Conjugate Diameter, Draw Two Right Lines, as CN , and Cn , Infinitely continued (as in the following Figure) and they will be the *Asymptotes* required.

That is, they are Two such Right Lines, as being Infinitely extended, will continually encline to the Sides of the Hyperbola, but never touch them.

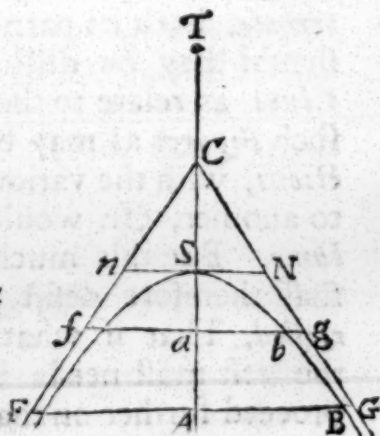
Demonstration.

Suppose the *Semi-ordinates* ab , and AB to be Rightly Applied to the Axis TA , and produced both ways to the *Asymptotes*, as at fg , and FG . Then will the $\triangle CSN$, $\triangle Cag$, and $\triangle CAG$, be alike.

Let $d = CS = TC$. And $L =$ the *Latus Rectum*, As before.

Put $\begin{cases} e = Sa \\ y = SA \end{cases}$ the *Abscissa's*. Then $\begin{cases} d + e = Ca \\ d + y = CA \end{cases}$.

Then	1	$d : SN :: d + e : ag$. Viz. $CS : SN :: Ca : ag$
1 in $\square$'s	2	$dd : \square SN :: dd + 2de + ee : \square ag$
But	3	$\frac{1}{2}dL = \square SN$ Per Sect. 3.
2, 3 $\therefore$	4	$\frac{ddL + 2deL + eeL}{2d} = \square ag$
Again	5	$2d : L :: 2de + ee : \square ab$ Per Sect. 2.
5 $\therefore$	6	$\frac{2deL + eeL}{2d} = \square ab$
4 $\rightarrow$ 6	7	$\frac{dL}{2} = \square ag - \square ab$
But $\begin{cases}$	8	$ag + ab = bf$
$\begin{cases}$	9	$ag - ab = bg$
8 $\times$ 9	10	$\square ag - \square ab = bf \times bg$
7, 10	11	$bf \times bg = \frac{1}{2}dL$
Again	12	$dd : \square SN :: dd + 2dy + yy : \square AG$
That is,		$\square CS : \square SN :: \square CA : \square AG$
3, 12 $\therefore$	13	$\frac{ddL + 2dyL + yyL}{2d} = \square AG$
But	14	$2d : L :: 2dy + yy : \square AB$ per Sect. 2.
14 $\therefore$	15	$\frac{2dyL + yyL}{2d} = \square AB$



13 — 15	16	$\frac{dL}{2} = \square AG - \square AB$	
Also {	17	$AG + AB = BF$	} per Figure.
	18	$AG - AB = BG$	
17 x 18	19	$\square AG - \square AB = BF \times BG$	
16, 19	20	$BF \times BG = \frac{1}{2}dL$	
11, & 20 ÷	21	$bg = \frac{\frac{1}{2}dL}{bf}$	And $BG = \frac{\frac{1}{2}dL}{BF}$

From the Last Step it is evident, that the *Assymptotes* are nearer the *Hyperbola* at *G*, than at *g*, and consequently will continually *Approach* to its *Curve*. For $BF) \frac{1}{2}dL (= BG$ is Less than $bf) \frac{1}{2}dL (= bg$, because the *Divisor* BF , is greater than the *Divisor* bf ; And it must needs be so, where ever the *Ordinates* are produced to the *Assymptotes*, from the Nature of the *Triangles*.

Again, from the 7th and 16th Steps it is as evident, that the *Assymptotes* can never really meet and be co-incident with the *Curve* of the *Hyperbola*, altho' both were *Infinitely* extended; because $\frac{1}{2}dL$ will always be the *Difference* between the *Square* of any *Semi-ordinate*, and the *Square* of that *Semi-ordinate* when it is produced to the *Assymptote*.

Confectary.

From hence it follows, that every *Right Line* which passes through the *Center*, and falls within the *Assymptotes*, will cut the *Hyperbola*; And all such *Lines* are called *Diameters*, as in the *Ellipsis*, because the *Properties* of the *Hyperbola* and *Ellipsis* are the same.

Note, Every *Diameter*, both in the *Ellipsis*, *Parabola*, and *Hyperbola*, hath its particular *Latus Rectum* and *Ordinates*; which, should they be distinctly handled, and the Effect of all the *Lines* as relate to them; As also, the Nature and *Properties* of such *Figures* as may be *Inscribed* and *Circumscribed* to all the *Sections*, with the various *Habitude* or *Proportions* of one *Hyperbola* to another, &c. would afford Matter sufficient to fill a large *Volume*. But this much may suffice by way of *Introduction*; I shall therefore desist pursuing them any further, being fully satisfied, That if what I have already done, be well understood, the rest must needs be very Easie to any One that Intends to proceed further on that Subject.

A N

INTRODUCTION TO THE Mathematicks.

PART V.

THE Method of finding out any particular *Quantity*, (viz. either any *Line*, *Superficies*, or *Solid*) by a regular *Progression* or *Series* of *Quantities*, continually approaching to it, and being *Ininitely* continued, would then become perfectly *Equal* to it; is, what is commonly called *Arithmetick* of *Infinities*; which I shall briefly deliver in the following *Lemma's*, and apply them to Practice in finding the *Superficial*, and *Solid Contents* of *Geometrical Figures* further on.

L E M M A I.

In any *Series* of *Equal Numbers* (representing *Lines* or other *Quantities*) As 1. 1. 1. 1. &c. Or 2. 2. 2. 2. &c. Or 3. 3. 3. 3. &c. If One of the *Terms* be Multiplied into the *Number* of *Terms*, the *Product* will be the *Sum* of all the *Terms* in the *Series*:

This is so very Plain and Easie to be understood, that it needs no *Example*.

L E M M A II.

If a *Series* of *Numbers* in *Arithmetick Progression*, begin with a *Cypher*, and the common *Difference* be 1. As 0. 1. 2. 3. 4. &c (representing a *Series* of *Lines* or *Roots* beginning with a *Point*) If the *Last Term* be Multiplied into the *Number* of *Terms*, the *Product* will be *Double* the *Sum* of all the *Series*.

That is, putting $L =$ the *Last Term*; $N =$ the *Number* of *Terms*; and $S =$ the *Sum* of all the *Series*.

Then

Then will $NL = 2S$. Consequently, $\frac{1}{2}NL = S$.
viz. one half of so many Times the greatest Term, as there are Number of Terms in the Series.

Thus $\frac{0+1+2+3+4}{4+4+4+4+4} = \frac{10}{20} = \frac{1}{2}$ the Sum of the Series $= \frac{1}{2}NL$.

And this will always be so, how many Terms soever there are, per Consect. 1. Page 185.

LEMMA III.

If a Series of Squares whose Sides or Roots are in Arithmetick Progression, beginning with a Cypher, &c. (as in the Last Lemma) be Infinitely continued; the Last Term being Multiplied into the Number of Terms, will be Triple to the Sum of all the Series, *viz.* $NLL = 3S$ Or $\frac{1}{3}NLL = S$.

That is, the Sum of such a Series will be one Third of the Last or Greatest Term, so many Times repeated as there are Number of Terms in the Series.

Instances in Square Numbers.

$$1. \frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}.$$

$$2. \frac{0+1+4+9}{9+9+9+9} = \frac{14}{36} = \frac{7}{18} = \frac{1}{3} + \frac{1}{18}$$

$$3. \frac{0+1+4+9+16}{16+16+16+16+16} = \frac{30}{80} = \frac{3}{8} = \frac{9}{24} = \frac{1}{3} + \frac{1}{24} \text{ \&c.}$$

From these Instances it's Evident, that as the Number of Terms in the Series do Increase; the Fraction or Excess above $\frac{1}{3}$ does Decrease, the said Excess always being $\frac{1}{6N-6}$. which if we suppose the Series to be Infinitely continued, will then become Infinitely small, *viz.* in Effect nothing at all.

Consequently $\frac{1}{3}NLL$ may be taken for the true or perfect Sum of such an Infinite Series of Squares.

LEMMA IV.

If a Series of Cubes, whose Roots are in Arithmetick Progression, beginning with a Cypher (&c. as above) be Infinitely continued; the Sum of all the Series will be $\frac{1}{4}NLLL = S$.

That is, one fourth of the Last or Greatest Term, so many Times repeated as there are Number of Terms,

Instances

Instances in Cube Numbers.

If 0 . 1 . 2 . 3 . &c. be the *Roots* of the *Cubes*,

Then 1. $\left\{ \frac{0 + 1 + 8 + 27}{27 + 27 + 27 + 27} = \frac{36}{108} = \frac{4}{12} = \frac{1}{4} + \frac{1}{12} \right.$

2. $\left\{ \frac{0 + 1 + 8 + 27 + 64}{64 + 64 + 64 + 64 + 64} = \frac{100}{320} = \frac{10}{32} = \frac{5}{16} = \frac{1}{4} + \frac{1}{16} \right.$

3. $\left\{ \frac{0 + 1 + 8 + 27 + 64 + 125}{125 + 125 + 125 + 125 + 125 + 125} = \frac{225}{750} = \frac{45}{150} = \frac{3}{10} = \frac{6}{20} \right.$
 $= \frac{1}{4} + \frac{1}{20}$

From these *Examples* it plainly appears, that as the *Number* of *Terms* in the *Series* *Increases*, the *Fraction* or *Excess* above $\frac{1}{4}$ *Decreases*; the *Excess* being always $\frac{1}{4N-4}$ which, if we suppose the *Series* to be *Ininitely continued*, will become *Infinite-ly small*, Or rather nothing. As in the *Last Lemma*.

Consequently $\frac{1}{4}NLLL$ may be taken for the true and perfect *Sum* of all the *Terms* in such an *Infinite Series* of *Cubes*.

LEMMA V.

If a *Series* of *Biquadrats*, whose *Roots* are in *Arithmetick Progression* beginning with a *Cypher*, &c. As before, be *Ininitely continued*, the *Sum* of all the *Terms* in such a *Series* will be $\frac{1}{3}NL^4$.

The *Truth* of this may be manifested by the like *Process*, as in the foregoing *Lemma's*, and so on for higher *Powers*. But if any one desire a further *Demonstration* of these *Series*, he may (I presume) meet with ample *Satisfaction* in *Dr. Wallis's History of Algebra*, Chap. 78 and 79, wherein the *Doctor* concludes with these *Words*.

" Thus having shewed, that in a *Progression* of *Laterals* (or *Arithmetical Proportionals*) beginning at 0. the *Sum* of 2. 3. 4. 5. 6 *Terms*, is always equal to $\frac{1}{2}$ of so many times the *Greatest* (and there being no *Pretence* of *Reason* why we should then doubt it, in a *Progression* of 7. 8. 9. 10. &c.) we conclude it so to be, though such *Number* of *Terms* be supposed *Infinite*.

" Again, in a *Progression* of their *Squares*; having shewed, that in 2. 3. 4. 5. 6 *Terms*, the *Aggregate* is always more than $\frac{1}{3}$ of so many times the *Greatest*; and the *Excess* always such *Aliquot Part* of the *Greatest*, as is denominated

E e e

by

“ by six times the Number of Terms wanting 1. (As if the
 “ Terms be 2, it is $\frac{1}{3} + \frac{1}{2}$; if 3, it is $\frac{1}{3} + \frac{1}{2} + \frac{1}{3}$; if 4, it
 “ is $\frac{1}{3} + \frac{1}{2} + \frac{1}{3}$; if 5, it is $\frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ of so many Times the
 “ greatest Term, and so onward) we may well conclude, (there
 “ being no Pretence of Reason why to doubt it in the rest)
 “ that it will be so how many soever be such Number of
 “ Terms. And because such Excess, as the Number of Terms
 “ do Increase, will become Infinitely small (or Less than any
 “ Assignable) we conclude (from the Method of Exhausti-
 “ ons) that, if the Number of Terms be supposed Infinite,
 “ such Excess must be supposed to vanish, and the Aggregate
 “ of such Infinite Progression supposed Equal to $\frac{1}{3}$ of so many
 “ times the greatest.

“ In like manner, having proved that such Progression of
 “ Cubes doth (as the Number of Terms increaseth) approach
 “ Infinitely near to $\frac{1}{4}$ of so many times the Greatest, and of
 “ Biquadrats to $\frac{1}{5}$, and so of Surfsolids to $\frac{1}{6}$ of so many times
 “ the Greatest, and so onwards as we please to try; (and there
 “ being no Pretence of Reason why to doubt it as to the rest;)
 “ we may take it as a sufficient Discovery, that (universally)
 “ the Aggregate of such Infinite Progression is Equal (or doth
 “ approach Infinitely near to such a Part of so many times the
 “ Greatest, as is denominated by the Exponent (or Number of
 “ Dimensions,) of such Power (as is that according to which
 “ the Progression is made,) Increased by 1. Namely, of Late-
 “ rals $\frac{1}{2}$; of Squares $\frac{1}{3}$; of Cubes $\frac{1}{4}$; of Biquadrats $\frac{1}{5}$
 “ (of so many times the Greatest;) and so onwards Infinitely.

This Discourse of the *Doctor's* I thought convenient to Insert,
 supposing it may give some Satisfaction to the Learner, to hear
 so great a Man as Dr. Wallis's Arguments about the Truth of
 these Series; which I have briefly delivered in the foregoing
Lemma's.

L E M M A VI.

If any Two Series or Ranks of Proportionals, have the same
 Number of Terms (whether Finite or Infinite) it will always

be { As the first Term of one Series : Is to the first Term of the
 other Series :: So is the Sum of all the Terms in the one Series
 To the Sum of all the Terms in the other Series. (12. e. 5.)

Again, in a Progression of their Squares; having shew'd
 that in 2, 3, 4, 5, 6 Terms the Aggregate is always more
 than 1 of so many times the Greatest; and the Excess al-
 ways less than $\frac{1}{2}$ of the Greatest, as is denominated

As in these Numbers,	1	3	Or these Numbers	4	5
	2	6		12	15
	3	9		36	45
	4	12		108	135
	5	15		324	405
	6	18		972	1215

That is, $1 : 3 :: 21 : 63$. And $4 : 5 :: 1456 : 1820$ &c.

The Application of these *Lemma's* to *Geometrical Quantities*, viz. to *Lines*, *Superficies*, and *Solids*, wholly depends upon granting the following *Hypotheses*.

The Hypotheses.

1. That every *Line* is supposed to consist or be composed of, an *Infinite Series* of *Equidistant Points*.

2. A *Surface* (viz. the *Area* of any *Figure*) to consist of an *Infinite Series* of *Lines*, either *Streight* or *Crooked*, according as the *Figure* requires.

3. A *Solid* to consist of an *Infinite Series* of *Plains* or *Superficies*, according as its *Figure* requires.

Not that we suppose *Lines*, which have really no *Breadth*, can fill a *Space* or *Superficies*; Or that *Plains* which have not any *Thickness*, can constitute a *Solid*: But by what we here call *Lines*, are to be understood small *Parallelograms* (or other *Superficies*) *Ininitely* Narrow; yet so as that their *Breadths*, being all taken and put together, must be *Equal* to the *Figure* they are supposed to fill up.

And those *Plains* or *Superficies*, which are here said to constitute a *Solid*, are to be understood *Ininitely Thin*; yet so as that their *Depths* or *Thicknesses* (which are hereafter also called *Lines*) being all taken together, must be *Equal* to the *Height* of the proposed *Solid*.

Now in order to render this *Hypothesis* as *Easie* for a *Learner* to understand as I can; I shall here propose a very plain and familiar *Example*.

Viz. Let us suppose any *Book* to be composed, or made up of 100. 200. 300 (more or Less) *Leaves* of *Fine Paper*; such a *Book* being close put together, will have *Length*, *Breadth*, and *Depth* or *Thickness*, and therefore may (not improperly) be called a *Solid*; and each of its *Edges* (being evenly Cut) will be a *Superficies* composed of a *Series* of small *Parallelograms*, every one of their *Breadths* being only the *Edge* of a single *Leaf* of *Paper*; And if we conceive the *Thickness* of every one of those *Leaves*, to

be *Divided* into 10. Or 100. Or 1000. &c. they will then become such a *Series* of *Infinitely* small *Lines*, as are (by the *Hypothesis*) said to compose, or fill up a *Superficies*.

And all the *Superficies* of those *Infinitely* Thin, or *Divided* *Leaves* of *Paper*, will become such a *Series* of *Plains* or *Superficies*, as are said to constitute a *Solid*; viz. such a *Solid* as the *Bigness* and *Figure* of that *Book*.

Now according to this *Idea* of *Lines*, *Superficies* and *Solids*, one may, without the least *Prejudice* to any *Demonstration*, admit of the following *Definitions* and *Theorems*.

Definitions.

I. The *Area's* of *Squares*, and all other *Parallelograms*, are composed or filled up, with an *Infinite Series* of *Equal Right Lines*.

II. The *Area* of every plain *Triangle*, is composed of an *Infinite Series* of *Right Lines* Parallel to its *Base*, and Equally decreasing until they terminate in a *Point* at the *Vertical Angle*.

III. The *Area* of a *Circle* may be composed either of an *Infinite Series* of *Concentrick* or *Parallel Circles*; Or of an *Infinite Series* of *Chord Lines* Parallel to its *Diameter*; Or of an *Innumerable* Multitude of *Sectors*.

IV. The *Area* of an *Ellipsis*, may be composed, either of an *Infinite Series* of *Ordinates* Rightly applied; Or of an *Infinite Series* of *Right Lines* Parallel to its *Transverse Diameter*.

V. The *Area's* of the *Parabola* and *Hyperbola*, are composed of an *Infinite Series* of *Ordinates*; Or may also be composed of *Right Lines* Parallel to their *Axes*, &c.

VI. A *Prism* is a *solid Body* contained or included, within several *Equal Parallelograms*, having its *Bases* or *Ends*, *Equal* and *alike*; And it's generally named according to the *Figure* of its *Base*. That is,

VII. A *Cube* (or *Solid* like a *Dye*) is a *Prism* bounded or included, within *Six* *Equal Square Plains*.

VIII. A *Parallelopipedon* is a *Prism* that hath its *Sides* bounded or included, within four *Equal Parallelograms*, and *Two Square Bases* or *Ends*.

IX. A *Cylinder* (or *Solid* like the *Rowling Stone* in a *Garden*) is only a *Round Prism*, having its *Bases* or *Ends* a perfect *Circle*.

X. The

X. The *Solidity* of every *Prism* is composed of an *Infinite Series* of equal *Plains*, Parallel and alike to that of its *Base*.

XI. A *Pyramid* is a *Solid* bounded or included within several plain *Triangles*, set upon any *Polygonius Base*, having their *Vertical Angles* all meeting together in a *Point*, called the *Vertex*; and takes its Name from the *Figure* of its *Base*: viz. if it have a *Square Base*, it's called a *Square Pyramid*; if a *Triangle Base*, it's called a *Triangular Pyramid*, &c.

XII. A *Cone* is only a *Round Pyramid*, which hath been already defined in Page 355 &c.

XIII. The *Solidity* of every *Pyramid* is composed or constituted, of an *Infinite Series* of *Plains*, Parallel and alike to that of its *Base*, Equally decreasing until they terminate in a *Point* at the *Vertex*.

XIV. A *Sphere* or *Globe* (viz. a *Ball*) is a *Solid* bounded or included within one regular *Superficies*, being formed or generated, by the *Rotation* of a *Semi-circle* about its *Diameter*; (called the *Axis* of the *Sphere*) And its *Solidity* is composed or constituted, of an *Infinite Series* of *Concentrick Circles*, whose *Diameters* are the *Chords* of that *Circle* by which it was formed.

XV. A *Spheroid* (or *Egg-like Figure*) is a *Solid* bounded with one regular *Superficies*, formed by the *Rotation* of a *Semi-ellipsis* about its *Transverse Diameter* (called the *Axis* of the *Spheroid*) and its *Solidity* is constituted of an *Infinite Series* of *Concentrick Circles*, whose *Diameters* are the *Ordinates* of that *Ellipsis* by which it was formed.

XVI. There is another sort of a *Solid* called an *Oblate Spheroid*, being formed by the *Rotation* of an *Ellipsis* about its *Conjugate Diameter*; And it's like to a flat *Turnep*.

XVII. If a *Semi-parabola* be turned about its *Axis*, it will form a *Solid* called a *Parabolick Conoid*, being composed or constituted of an *Infinite Series* of *Circles*, whose *Diameters* are the *Ordinates* of the *Parabola*.

XVIII. If a *Parabola* be turned about its *Base* or greatest *Ordinate*, it will form a *Solid* called a *Pyramidoid*, but most commonly a *Parabolick Spindle*, which will be constituted of an *Infinite Series* of *Circles*, whose *Diameters* are *Right Lines* Parallel to the *Parabola's Axis*.

XIX. If an *Hyperbola* be turned about its *Axis*, it will form a *Solid* called a *Hyperbolick Conoid*, being constituted of an *Infinite Series* of *Circles*, whose *Diameters* are the *Ordinates* of the *Hyperbola*.

XX. The

XX. The Curve Superficies of all circular Solids (viz. Cylinders, Cones, Spheres, &c.) are composed of an Infinite Series of the Peripheries of those Circles which constitute their Solidities.

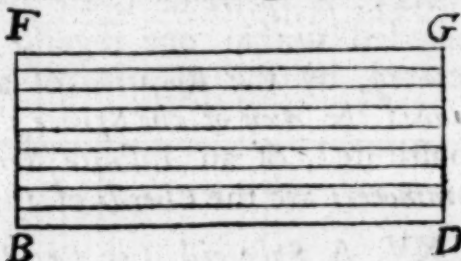
Upon these Definitions are grounded all the following Theorems; and therefore, if they were diligently compared with their respective Figures, it must needs be of great Help to the Learner, and would render all that follows very Easie; wherein I shall begin with what hath been already Demonstrated, by way of Introducing the rest.

THEOREM I.

The Area of every Right-lined Parallelogram, is obtained by Multiplying the Length into its Breadth.

That is, $BD \times FB =$ the Area of the Parallelogram BDFG. Per Lemma 1. compared with Definition 1.

Example. Suppose $BD = 26$.
And $FB = 9$.
Then $26 \times 9 = 234$. the Area.
See Problem 1. Page 333.



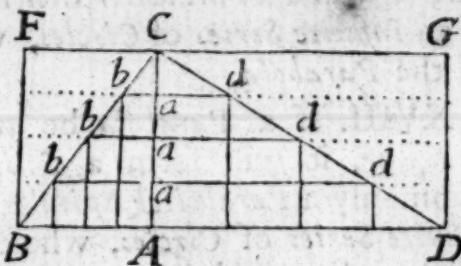
THEOREM II.

The Area of every Plain Triangle, is Equal to Half the Area of its Circumscribing Parallelogram.

That is, $\frac{BD \times CA}{2} =$ the Area of $\triangle BCD$, in the following Figure.

Demonstration.

Suppose the Perpendicular CA to be Divided into an Infinite Number of Equal Parts; as at the Points a, a, a , &c. and through those Points there were drawn Right Lines Parallel to the Base BD , viz. bad, bad, bad , &c. Then will those Lines be a Series of Terms in Arithmetick Progression, beginning at the Point C , viz. $0, bd, 2bd, 3bd$, &c. as is evident by the Figure, wherein BD is the Greatest Term $= L$, and CA the Number of Terms $= N$.



But

But $\frac{1}{2}NL = S$, per Lemma 2. And $S =$ the Triangle's Area, per Definition 2. Q. E. D.

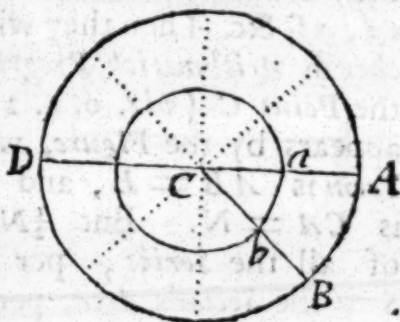
Example. Let $BD = 26$. and $CA = 9$. As above. Then $\frac{26 \times 9}{2} = 117$. Or $\frac{26}{2} \times 9 = 117$. Or thus $26 \times \frac{9}{2} = 117$. the Area required. See Problem 3. Page 324.

THEOREM III.

The Peripheries of Circles are in Proportion one to another, as their Diameters are.

Demonstration.

Let the Periphery of a Circle be Divided into any Number of Equal Arches, by Right Lines drawn from the Center (viz. Radius's) suppose them 8, as in the Annexed Figure, wherein AB is one of them. Then if through any Point in the Radius, there be drawn a Concentrick or Parallel Circle, its Periphery will also be Divided into 8 Equal Arches by those Radius's, one whereof will be ab ; And the $\triangle Cab$ will be like to $\triangle CAB$.



Therefore $Ca : ab :: CA : AB$. Or $Ca : CA :: ab : AB$. Consequently $2Ca : 2CA :: 8ab : 8AB$.

But $2Ca = da$, the Diameter of the Circle whose Periphery is $8ab$.

And $2CA = DA$, the Diameter of the Circle whose Periphery is $8AB$. Therefore, &c. as per Theorem. Q. E. D.

Example. In Chapter 6, Part 3. it was found, that if the Diameter of a Circle be 2. its Periphery will be 6,2831853071795864. Ergo $2 : 6,2831853071795864 :: 1 : 3,1415926535897932$ the Periphery of the Circle whose Diameter is 1.

Corollary.

Hence it follows, that because Unity or 1. may be made the First Term in the Proportion, therefore 3,14159265 &c. may be made a constant or settled Factor, which being Multiplied into any proposed Diameter, will produce the Periphery of that Circle.

Note, Instead of 3,14159265 &c. it may be sufficient to take 3,1416. Or

Or in whole Numbers the Proportion may be
As $7 : 22 :: \text{Diam.} : \text{Periphery}$ } { these Numbers may serve, and
 Or $113 : 355 :: \text{Diam.} : \text{Periphery}$ } { are often used in com. Practice

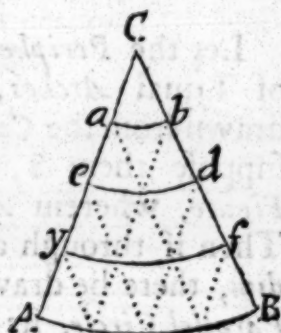
THEOREM IV.

The Area of any Sector of a Circle, is Equal to Half the Rectangle of the Radius into its Arch.

That is, $\frac{CA \times AB}{2} = \text{the Area of } ACB.$

Demonstration.

Suppose the Radius CA to be divided into an Infinite Series of Equidistant Points; as a, e, y , &c. and through those Points there were drawn Concentrick or Parallel Arches, as ab, ed, yf , &c. Then they will be a Series of Arches in Arithmetick Progression, beginning at the Point C (*viz.* 0. 1. 2. 3. &c.) as plainly appears by the Figure, wherein the Greatest Term is $AB = L$, and Number of Terms is $CA = N$. But $\frac{1}{2}NL = S$ the Sum of all the Series, per Lemma 2. And $S = \text{the Sector's Area}$, per Definition 3. Q. E. D.



Example. Let the Radius $CA = 12$. And the Arch $AB = 8$. Then $\frac{12 \times 8}{2} = 48$. Or $\frac{1}{2} \times 8 \times 12 = 48$. Or $\frac{1}{2} \times 12 \times 8 = 48$. the Area of the Sector ACB .

THEOREM V.

The Area of every Circle, is Equal to Half the Rectangle of the Radius into its Periphery.

That is, according to Archimedes, a Circle is Equal to a Right-angled Triangle, whose Sides containing the Right Angle, are Equal, one to the Radius, and the other to the Perimeter of that Circle. Pro 1. de Dimensione Circuli.

The Truth of this Theorem may be easily deduced from the Last; Thus, if we suppose the Last Sector to be one eight Part of a Circle; then it follows, that $\frac{8AB \times CA}{2} = 4AB \times CA$ will be the Area of the whole Circle.

But $4AB = \text{Half the Circle's Periphery}$; And $CA = \text{Half its Diameter}$. Therefore, &c. as per Theorem. Q. E. D. But

Example. If the *Diameter* be *Unity* or 1. the *Periphery* will be 3,14159265 &c. per *Theorem 3*.

Then $\frac{3,14159265}{2} \times \frac{1}{2} = 0,78539816 \text{ \&c.}$ (Or 0,7854 for common Use) will be the *Area* of that *Circle*.

Scholium.

From hence naturally flows the following *Proportion* between the *Square* and its *Inscribed Circle*.

Proposition { *As the Perimeter (viz. the Sum of the four Sides) of any Square : Is to its Area :: So is the Periphery of the Inscribed Circle : To its Area.*

That is, supposing $AB = D =$ the *Side* of the *Square* and the *Diameter* of its *Inscribed Circle*.

Then $4D =$ the *Perimeter*; $DD =$ the *Area* of the *Square*: And $3,1416D =$ the *Periphery* of the *Circle*. Per *Theorem 3*.

But $4D : DD :: 3,1416D : 0,7854DD =$ the *Circle's Area*.

And if $D = 1$. Then $4D = 4$ and $DD = 1 \times 1 = 1$. And the *Periphery* will be 3,1416.

Then $4 : 1 :: 1 : 0,7854 \text{ \&c.}$ As in the *Example* above.



And from hence may be easily deduced the following *Theorem*:

THEOREM VI.

The *Area's* of all *Circles* are in *Proportion* one to another; As the *Squares* of their *Diameters*. (2. c. 12.)

For if $D =$ the *Diameter* of one *Circle*, and $d =$ the *Diameter* of another *Circle*.

Then will $0,7854DD$ be the *Area* of one *Circle*, and $0,7854dd$ will be the *Area* of the other *Circle*. As above.

But $0,7854DD : 0,7854dd :: DD : dd$. Or thus.

Let $D =$ the *Diameter*, and $P =$ the *Periphery* of one *Circle*. $d =$ the *Diameter*, and $p =$ the *Periphery* of another *Circle*.

Then	1	$\frac{1}{2}D \times \frac{1}{2}P = \frac{1}{4}DP = A$	the <i>Area</i> of one <i>Circle</i>
And	2	$\frac{1}{2}d \times \frac{1}{2}p = \frac{1}{4}dp = a$	the <i>Area</i> of the other <i>Circle</i> . Per
$1 \times \frac{4}{4}$	3	$DP = 4A$	(last <i>Theorem</i> ;
$2 \times \frac{4}{4}$	4	$dp = 4a$	
$3 \div D$	5	$P = \frac{4A}{D}$	

fff

$4 \div d$

4 ÷ d	6	$p = \frac{4a}{d}$	
But	7	$P : p :: D : d.$	Per Theorem 3.
5, 6, 7	8	$D : d :: \frac{4A}{D} : \frac{4a}{d}$	
8	9	$4DDa = 4dda.$	That is, $DDa = dda$
9, Anal.	10	$DD : A :: dd : a.$	Or $A : a :: DD : dd$ Q. E. D.

Corollary.

Hence it follows, that because the Square of 1. is 1. (viz. $1 \times 1 = 1$) and 0,78539816 &c. Or 0,7854 is the Area of the Circle whose Diameter is 1. (As before) therefore it will be $1 : 0,7854 ::$ So is the Square of any Circle's Diameter : To its Area. And because 1. is the First Term in the Proportion, therefore 0,7854 may be made a constant Factor ; which being Multiplied into the Square of any proposed Diameter, will produce the Area of that Circle.

Note, The Four last Theorems do plainly show the Reason of all the common or Practical Problems about a Circle, which for the Learner's further Satisfaction, I have here Inserted together ; Supposing as before,

That $\begin{cases} D = \text{the Diameter} \\ P = \text{the Periphery} \\ A = \text{the Area} \end{cases}$ of any proposed Circle.

Then	Prob. 1. D, being given ; To find P.
1	1. $1 : 3,1416 :: D : P.$ Per Theorem 3.
Example	2. $3,1416D = P$ Suppose = 32. Then $3,1416 \times 32 = 100,5312$ the Periphery.
	Prob. 2. D, being given ; To find A.
3	3. $1 : 0,7854 :: DD : A.$ Per Theorem 6.
Example	4. $0,7854DD = A$ Suppose D = 32. As before, Then $DD = 32 \times 32 = 1024.$ And $0,7854 \times 1024 = 804,2496$ the Area required.
	Prob. 3. P, being given ; To find D. $\times 1$
2 ÷	5. $D = \frac{P}{3,1416}$ } Or { because $\frac{1}{3,1416} = 0,3183$ This being only Converse to the First needs no Exam.

			<i>Prob. 4. P, being given ; To find A.</i>	
2 ⊙ 2	6	9,86965	$DD = PP$	
6 ÷	7	$DD = \frac{PP}{9,86965}$	Or	$0,10132PP = DD$
4 ÷	8	$DD = \frac{A}{0,7854}$	Or	$1,2732A = DD$
For		$\frac{1}{0,7854} = 1,2732$		
7, 8	9	$\frac{PP}{9,86965} = \frac{A}{0,7854}$	Or	$0,10132PP = 1,2732A$
9 × &c.	10	$\frac{PP}{12,5664} = A$	Or	$0,07957PP = A$
			<i>Prob. 5. A, being given ; To find D.</i>	
8 uw 2	11	$D = \sqrt{\frac{A}{0,7854}}$	Or	$D = \sqrt{1,2732A}$
			<i>Prob. 6. A, being given ; To find P.</i>	
10 × &c.	12	$PP = 12,5664A$	Or	$PP = \frac{A}{0,07957}$
12 uw 2	13	$P = \sqrt{12,5664A}$	Or	$P = \sqrt{\frac{A}{0,07957}}$

These *Six Problems* contain all the *Variety* that can be proposed about finding the *Periphery*, *Diameter*, and *Area* of any *Circle*.

But if it be required to find the *Area* of any *Segment*, or Part of a *Circle* cut off by a *Chord*, that *Work* will require a further *Consideration*.

First, As to the *Data*, there must always be given the *Diameter* ; Or, either the *Periphery*, or *Area* of the *Circle*, in order to find the *Diameter*.

Secondly, There must also be given, either the *Chord*, which is the *Base* of the *Segment* ; Or the *Versed Sine*, which is the *Height* of the *Segment*.

That is, either *BG*, or *AF*, in the following *Scheme*, must be given, that so the *Area* of the $\triangle BCG$ may be found.

Then it's evident (*by the Figure*) that if the *Area* of the $\triangle BCG$ be taken from the *Area* of the *Sector CBAG*, the *Remainder* will be the *Area* of the *Segment BAG*.

And if the *Area* of the *Segment BAG*, be taken from the whole *Area* of the *Circle* ; the *Remainder* will be the *Area* of the other *Segment DBG*.

Example in Numbers.

Let there be given $DA = 32$, as in *Prob. 1*.

And the *versed Sine* $AF = 6$.

Then $\frac{1}{2}DA = BC = CA = 16$.

And $CA - AF = CF = 10$.

But $\square BC - \square CF = \square BF$.

Consequently $\sqrt{\square BC - \square CF} = BF$.

Viz. $\sqrt{156} = 12,49 = BF$.

Then by the Doctrine of plain Triangles, the Arch $BA = \angle BCA$ may be found in Degrees and Decimal Parts;

Thus $BC : \text{Radius} :: BF : \text{Sine } \angle BCF = 51^{\circ}, 31 \text{ Degrees}$.
And then it will always hold in this Proportion.

Viz. $\left\{ \begin{array}{l} \text{As the Circle's Periphery in Degrees : Is to its Periphery} \\ \text{in Equal Parts (according to the Dimensions taken) : :} \\ \text{So is the Arch in Degrees (viz. } \angle BCA \text{) : To the same} \\ \text{Arch in Equal Parts.} \end{array} \right.$

That is, $360^{\circ} : 100,5312 :: 51^{\circ}, 31 : 14,3284 = BA$.

Then $14,3284 \times 16 = 229,2544$ the Area of the Sector $BCAG$.

And $12,49 \times 10 = 124,9$ the Area of the $\triangle BCG$.

Their Difference $104,3544 =$ the Area of the Segm. BAG .

Or the Area of any Segment may be otherwise found, (as most usually it is) by a Table of the Segments of a Circle whose Area is Unity or 1. The Construction or making of such a Table, is very well laid down in Mr. Darie's Book of Gauging, Chap. 9. which he performs in this Problem.

Problem.

In a Circle whose Area is Unity, and its Diameter Cut by Chord Lines into 1000 Equal Parts; To find the Segment to any *versed Sine* proposed, not exceeding 500 of those Equal Parts.

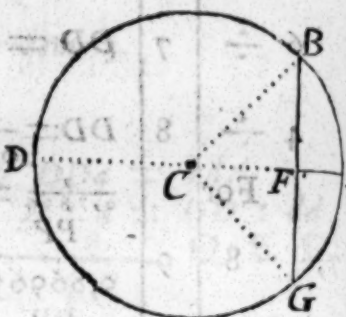
1. Multiply the *versed Sine* proposed, by 0,002. and Subtract the Product from an Unit or 1.

2. This Remainder you shall seek in the common Table of Natural Sines (the Arch being Divided into Degrees and Centesimal) which being found let its Co-Arch be Doubled, and called A .

3. You shall find the correspondent Sine to A , which Sine being found you may call S ; And then it holds.

$6,2831853 \times 0,0174532925 A = S$ ($=$ the Segment required.)

No



Now this *Segment* being thus found, if you *subduct* it from an *Unit*, you have the *Co-Segment*, &c.

Note. Notwithstanding what hath been said in the second Precept of this Problem, it very often falls out, that the Remainder there spoken of, cannot be truly found in the Table of Natural Sines; therefore, in this Case my Advice is, That you make Two Operations, One with a Sine the next Greater, and One with a Sine the next Less; and in so doing you will be sure to have the Segment required bounded between the Results of those Two Operations.

Example. Let it be proposed to find the Correspondent Segment to the Versed Sine 263.

First $263 \times 0,002 = 0,526$. and $1 - 0,526 = 0,474$. its Arch is $28^{\circ},29$ being Less than just; its Complement is $61,71$ which being doubled, is $123,42 = A$.

Then $,0174533A = 2,154086286$

$- 0,8346556 = S$ The Sine of A .

$6,2831853) 1,319430686 (0,209993$ the Segment.

Now I make a second Work.

263 being Multiplied with 0,002 is 526. and $1 - 526 = 0,474$ its Arch is $28^{\circ},30$ being Greater than just; And its Complement is $61,70$ which being doubled, is $123,4 = A$.

Then $,0174533A = 2,1537372$

$- 0,8348478 = S$ The Sine of A .

$6,2831853) 1,3188894 (0,209907$ The Segment.

So you see by these Two Operations, that the Segment is bounded, and 'tis very probable it may be 0,20995.

But to abbreviate this Large Factor, and this Large Divisor, I shall here Insert Two Tablets of them, which will be ready for use and Exact enough too.

Divisor.		Factor.	
6,2832	1	,0174533	1
12,5664	2	,0349066	2
18,8495	3	,0523599	3
25,1327	4	,0698132	4
31,4159	5	,0872665	5
37,6991	6	,1047197	6
43,9823	7	,1221730	7
50,2655	8	,1396263	8
56,5487	9	,1570796	9

Thus far Mr. Dary; which I have here Inserted to shew the Learner, how by the Help of these Two Tablets, and a Table of Natural Sines, he may Easily make a Table of Segments, whose Use shall be shewed farther on; viz. when I come to treat of practical Gauging. In the mean time I shall here lay down another Method,
To

To find the Area of any Segment of a Circle (*very near*) by a New Theorem, without the Help either of a Table of Sines or Segments, having the same Data as before in Page 404.

Viz. Let $\begin{cases} R = \text{the Radius, or } \frac{1}{2} \text{ Diameter of the given Circle.} \\ d = \text{the Difference between the Versed Sine and Radius.} \\ C = \text{Half the Chord of the Segment's Base.} \end{cases}$

Theorem $\left\{ \frac{2\frac{1}{3}RR - 1\frac{1}{3}Rd - dd}{1\frac{1}{2}R + d} \times C = S, \text{ the Area of the Segm.} \right.$

Example. Suppose $R = BC = 16$. $d = FC = 10$. and $C = BF = 12,49$. As before.

Then $2\frac{1}{3}RR = 597,3333$. $1\frac{1}{3}Rd = 213,3333$. $dd = 100$
 $- 313,3333 = 1\frac{1}{3}Rd + dd$

$1\frac{1}{2}R + d = 34$ 284,0000 (8,3529

Lastly $8,3529 \times 12,49 = 104,3276$ the Area of the Segment BAG, As before.

THEOREM VII.

As Squares are to the Area's of their Inscribed Circles; So are Parallelograms to the Area's of their Inscribed Ellipsis.

That is, $\left\{ \begin{array}{l} \text{As the Square of the Diameter of any Circle : Is to its} \\ \text{Area} :: \text{So is the Rectangle of the Transverse and Conju-} \\ \text{gate Diameters of any Ellipsis : To its Area.} \end{array} \right.$

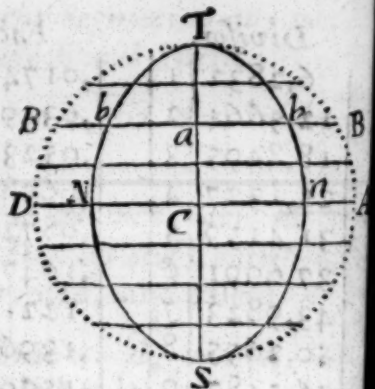
Demonstration.

Circumscribe any Ellipsis with a Circle; and suppose an Infinite Number of Chord Lines drawn therein, all Parallel to the Conjugate Diameter, as those in the annexed Figure; Then it will

be $\left\{ \begin{array}{l} \text{As (DA) the Diameter of the Circle : Is to (Nn) the} \\ \text{Conjugate Diameter of the Ellipsis} :: \text{So is (B a B) any} \\ \text{Chord in the Circle : To (b a b) its respective Ordi-} \\ \text{nate in the Ellipsis.} \end{array} \right.$

For according to the Property of the Circle

it is	1	$TS - Ta \times Ta = \square Ba$
And		by the Property of the Ellipsis
it is	2	$\square TC : \square NC :: TS - Ta \times Ta : \square ba$
1, 2	3	$\square TC : \square NC :: \square BA : \square ba$
3, hence	4	$TC : NC :: Ba : ba$
Conseq.	5	$2TC : 2NC :: 2Ba : 2ba$
That is,	6	$DA : Nn :: B a B : b a b$
Put	7	$D = 2TC$, and $d = 2NC$
Then	8	$D : d :: \text{Chord } B a B : \text{Ordinate } b a b, \&c.$



But

But the *Sum* of an *Infinite Series* of such *Chords*, as BaB , do constitute the *Area* of the *Circle*, per *Definition* 3.

And the *Sum* of the like *Series* of their respective *Ordinates*, as bab , do constitute the *Ellipsis Area*, per *Definition* 4.

Therefore $D : d :: \text{Circle's Area} : \text{Ellipsis Area}$, per *Lemma* 6.

But $D : d :: DD : Dd$. Whence it follows, That $DD : \text{Circle's Area} :: Dd : \text{Ellipsis Area}$. Q. E. D.

Consequently, As 1 : Is to 0,7854 :: So is the *Rectangle* or *Product* of the *Transverse* and *Conjugate Diameters* of any *Ellipsis* : To its *Area*.

Example. Suppose $TS = 36$. and $Nn = 16$. Then $36 \times 16 = 576$ And $576 \times 0,7854 = 452,3904$ the *Area* of the *Ellipsis*.

Corollaries.

1. Hence it is *Easie* to conceive, that the *Square Root* of the *Rectangle* or *Product* of the *Transverse* and *Conjugate Diameters*; will be the *Diameter* of a *Circle*, whose *Area* will be *Equal* to the *Ellipsis Area*.

Viz. $\sqrt{576} = 24$ the *Diameter* of a *Circle* = to the *Ellipsis*.

2. All *Segments* of an *Ellipsis* and its *Circumscribing Circle*, (whose *Bases* are *Parallel* to the *Conjugate Diameter*, and of the *same Height*) are in *Proportion* one to another, as their *Bases* are.

That is, $BaB : bab :: \text{Area Segment } BTB : \text{Area Segment } bTb$ Or $TS : Nn :: \text{Area Segment } BTB : \text{Area Segment } bTb$.

T H E O R E M VIII.

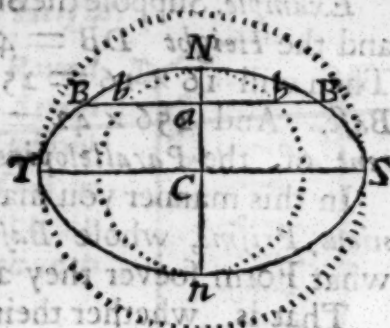
The *Area* of every *Ellipsis* is a *mean Proportional*, between the *Area's* of its *Circumscribing* and *Inscribed Circles*.

The *Truth* of this *Theorem* may be easily deduced from the *Last*; For supposing $D = TS$, and $d = Nn$, as before. Then it is already proved, that $DD : Dd :: \text{Circumscribing Circle's Area} : \text{Ellipsis Area}$.

But $DD : Dd :: Dd : dd$.

Therefore, $\text{Ellipsis Area} : \text{Inscribed Circle's Area} :: Dd : dd$.

Per *Theorem* 6.



Example. Let $TS = D = 36$. and $Nn = d = 16$. As before. Then $DD = 1296$. And $dd = 256$.

Then

Then will $\begin{cases} 1296 \times 0,7854 = 1017,8784 \text{ the Great Circle's Area} \\ 256 \times 0,7854 = 201,0624 \text{ the Lesser Circle's Area} \end{cases}$

Suppose $A =$ the Ellipsis Area. Then according to the Theorem, it will be, $1017,8784 : A :: A : 201,0624$.

Ergo $AA = 1017,8784 \times 201,0624 = 204657,07401216$

Consequently, $\sqrt{204657,07401216} = 452,3904 = A$, the Area of the Ellipsis, as before in the Last Example.

Cerollary.

From hence it follows, that all Segments of an Ellipsis, and its Inscribed Circle (whose Bases are Parallel to the Transverse Diameter, and have the same Height) are in Proportion one to another, as the Area's of the Ellipsis and Circle are.

That is, Area of Circle : Area of Ellipsis :: Segment bNb : Segment BNB .

Or, $Nn : TS ::$ Area Segment bNb : Area Segment BNB .

THEOREM IX.

The Solid Content of any Prism (what Figure soever its Base is of) is obtain'd by Multiplying the Area of its Base into its Height.

For Instance, a Parallelopipedon (or Square Prism) is constituted of an Infinite Series of Equal Squares: that of its Base $BAb a$ being one of the Terms, and its Height DB or GA the Number of all the Terms.

Consequently, the Area of $BAb a \times DB =$ the Sum of all the Series (per Lemma 1.) which is the Solidity of the Parallelopipedon $DBG A$, per Definition 10.

Example. Suppose the Side of the Base $BA = 16$ and the Height $DB = 42$. Then will $16 \times 16 = 256$ be the Area of the Base. And $256 \times 42 = 10752$ the Solid Content of the Parallelopipedon $DBG A$.

In this manner you may find the Solidity of all regular Polygonous Prisms, whose Bases (or Ends) are Parallel and alike; what Form soever they are of.

That is, whether their Bases are Triangles, Pentagons, Hexagons, Or Octagons, &c.



THEOREM X.

Every Pyramid is the Third Part of the Prism, that hath the same Base and Height with it. (7. e. 12.)

That is, The Solid Content of the Pyramid BVA , (in the last Figure) is one Third of its Circumscribing Prism $DBG A$.

Demonstration.

For every Pyramid that hath a Square Base (as $B A b a$ in the last Figure) is constituted of an Infinite Series of Squares, whose Sides or Roots, are continually increasing in Arithmetick Progression, beginning at the Vertex or Point V , (See Theor. 2.) its Base $B A b a$ being the Greatest Term ($= LL$) and its Perpendicular Height VC or DB , is the Number of all the Terms ($= N$); But $\frac{NLL}{3} = S$ the Sum of all the Series, per Lemma 3. and $S =$ the Solid Content of the Pyramid BVA , per Definition 13.

Example. Suppose the Side of a Pyramid's Base be $BA = 16$. and its Height be $VC = 42$. Then $16 \times 16 = 256$ the Area of its Base $B A b a$. And $\frac{256 \times 42}{3} = 3584$. Or $\frac{256}{3} \times 42 = 3584$. Or thus $256 \times \frac{42}{3} = 3584$ is the Solidity of that Pyramid BVA .

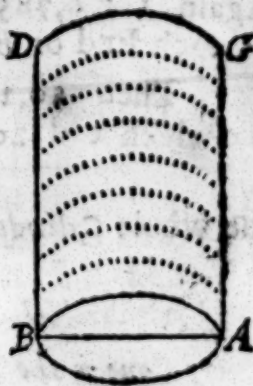
Corollary.

From hence it will be easy to conceive, That every Pyramid is $\frac{1}{3}$ of its Circumscribing Prism, what Form soever its Base is of; viz. whether it be a Square, Triangle, or Pentagon, &c.

THEOREM XI.

The Solid Content of every Cylinder, is obtained by Multiplying the Area of its Base into its Height.

For every Right Cylinder is only a Round Prism, being constituted of an Infinite Series of Equal Circles; that of its Base or End being one of the Terms, and its Height DB is the Number of all the Terms. Therefore the Area of its Base BA , being Multiplied into DB , will be its Solidity, per Lemma 1. Viz, Let $D = BA$, and $H = GA$. Then $0,7854DD \times H =$ its Solidity.



Example. Let the *Diameter* of its *Base* be $D = 16$, and its *Height* $H = 42$.

Then $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$, the *Area* of its *Base*.

And $201,0624 \times 42 = 8444,6208$ the *Solid Content* of that *Cylinder* $DBG A$.

Corollary.

Hence it's evident, That every *Square Parallelopipedon* Is to its *Inscribed Cylinder*, As 1 : Is to $0,7854$; Or in whole Numbers, As 452 : To 355 , very near.

And that all *Prisms* are in *Proportion* to their *Inscribed Cylinders*, As the *Area's* of their *Bases* are.

T H E O R E M XII.

The *Curve Superficies* of every *Right Cylinder*, is *Equal* to the *Rectangle* made of its *Height* into the *Periphery* of its *Base*.

That is, DB Multiplied into the *Periphery* of *Diameter* BA , will produce the *Curve Superficies* of the *Last Cylinder* $DBG A$.

For the *Cylinder* is *Constituted* of an *Infinite Series* of *Equal Circles* (according to the *Last Theor.*) therefore its *Curve Superficies* is composed of the *Peripheries* of those *Circles*, per *Definition* 20. But the *Periphery* of its *Base* BA is one of the *Terms*, and its *Height* DB is the *Number* of *Terms*. Therefore, &c. as per *Lemma* 1.

To which, if there be added the *Area's* of both its *Ends* (or *Bases*) the *Sum* will be the *Superficies* of the whole *Cylinder*.

Example. Suppose the *Diameter* of its *Base* to be $BA = 16$, and its *Height* $DB = 42$. As before.

Then $1 : 3,1416 :: 16 : 50,2656$ the *Periphery* of its *Base*.

Again, $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$, the *Area* of each *End* or *Base*.

Then $50,2656 \times 42 = 2111,1552$ the *Curve Superficies*.
To which add $201,0624 \times 2 = 402,1248$ both the *end Area's*.

The *Sum* = $2513,2800$ is the *Superficies* of the whole *Cylinder*.

T H E O R E M XIII.

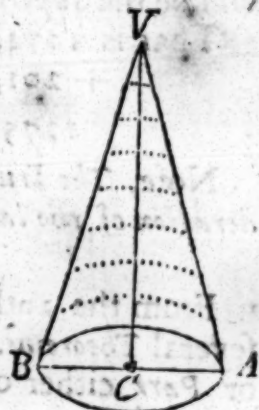
Every *Cone* is the *Third Part* of a *Cylinder*, having the same *Base* with it, and their *Altitudes* *Equal*. (10. e. 12.)

Demoustr.

Demonstration.

The Truth of this Theorem may be easily conceived by only considering, that a Cone is but a Round Pyramid, and therefore it must needs have the same Ratio to its Circumscribing Cylinder, as the Square Pyramid hath to its Circumscribing Parallelopipedon; viz. As 1 : To 3. However to make it yet clearer, let it be further considered, That

Every Right Cone is constituted of an Infinite Series of Circles, whose Diameters do continually Increase in Arithmetick Progression, beginning at the Vertex or Point *V*, the Area of its Base *BA* being the Greatest Term, and its Perpendicular Height *VC* the Number of all the Terms; therefore Area Circle *BA* $\times \frac{1}{3}VC$ will be the Sum of all the Series, per Lemma 3. which is the Cone's Solidity.



Example. Let the Diameter of its Base be $BA = 16$, and its Height $VC = 42$. Then $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$ the Area of the Base. And $\frac{201,0624 \times 42}{3} = 2814,8736$ the Solidity of the Cone *BVA*. Or thus, $201,0624 \times \frac{42}{3} = 2814,8736$ &c.

Corollary.

Hence it follows, That every Square Pyramid is to its Inscribed Cone, As 1 : 0,7854. (Or, As 452 : 355.) Consequently, that all Pyramids have the same Ratio to their Inscribed Cones, As the Area's of their Bases have.

THEOREM XIV.

The Curve Superficies of every Right Cone, is Equal to Half the Rectangle of the Periphery of its Base into the Length of its Side.

The Truth of this Theorem is self-evident from the Definition of a Cone, Chap. 1. Part 4. where it appears, that the Curve Superficies of every Right Cone (as *BVA*) is Equal to the Area of a Sector of that Circle, whose Radius is the Side of the Cone (*VB*) and its Arch Equal to the Periphery of the Cone's Base (*BA*.) But the Area of any Sector, is Equal to Half the Rectangle of the Radius into its Arch, per Theor. 4. Therefore, &c.

Example. Suppose the Length of the Cone's Side to be VB , or $VA = 42,7551$.

And the Diameter of its Base, viz. $BA = 16$. As before.

Then will 50,2656 be the Periphery of its Base.

And $\frac{50,2656 \times 42,7551}{3} = 1074,5553$ &c. the Curve of the Superficies.

To which, if there be added the Area of its Base, the Sum will be the Superficies of the whole (viz. all the) Cone.

That is 1074,5553

+ 201,0624 the Area of the Base.

1275,6177 the total Superficies &c.

Note, The Truth of this Theorem may be proved from the Consideration of the last Theorem, and Definition 20.

Scholium.

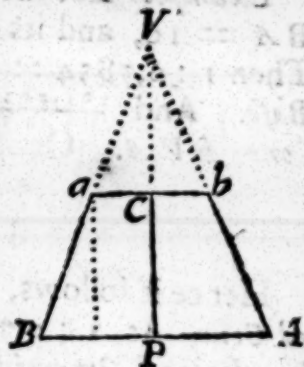
From the 10th and 13th Theorems, may be easily deduced several Theorems. for finding the Solid Content of any Frustrum or Part either of a Pyramid, or Cone, cut by a Plain Parallel to its Base.

Suppose a Square Pyramid, as BVA , to be cut by a Plain at ab Parallel to its Base BA , and it were required to find the Solidity of the Frustrum or Part $abAB$. Let there be given

$D = BA$ the Side of the Greater Base.

$d = ba$ the Side of the Lesser Base.

$H = CP$ the Perpendicular Height.



First 1 $D - d : H :: d : \frac{dH}{D - d} = VC$ by the Figure.

Then 2 $\left\{ \begin{array}{l} DD \times \frac{H + VC}{3} = \text{the whole Pyramid } BVA. \\ \text{Per Theorem 10.} \end{array} \right.$

And 3 $dd \times \frac{1}{3} VC = \text{the Pyramid } aVb \text{ cut off.}$

Viz. 1, 2 4 $\frac{DDDH}{3D - 3d} = \text{the whole Pyramid } BVA.$

And 1, 3 5 $\frac{dddH}{3D - 3d} = \text{the Pyramid } aVb.$

4 — 5 6 $\frac{DDDH - dddH}{3D - 3d} = \text{the Frustrum } abAB.$

6, Reduc. 7 $DD + Dd + dd : \times \frac{1}{3} H = \text{the Frustrum } abAB.$

Which in Words gives this following Theorem.

THE

Frustrum

T H E O R E M XV.

To the *Rectangle* of the *Sides* of the *Two Bases*, Add the *Sum* of their *Squares*; that *Sum* being *Multiplied* into one *Third* of the *Frustum's Height*, will give its *Solidity*.

Example. Suppose the *Side* of the *Greater Base* $BA = 16$. And the *Side* of the *Lesser Base* (or *Top*) $ab = 12$. The *Height* $CP = 9$.

Then $16 \times 12 = 192$. $16 \times 16 = 256$. and $12 \times 12 = 144$. Next $192 + 256 + 144 = 592$. And $\frac{592 \times 9}{3} = 1776$. Or $592 \times \frac{9}{3} = 1776$ the *Content* of the *Frustum* of a *Square Pyramid*.

And if it were the like *Frustum* of a *Right Cone*, it may be found by the same *Theorem*. Supposing $D =$ the *Diameter* of the *Greater Base*, $d =$ the *Diameter* of the *Lesser*, and $H =$ the *Height* of the *Frustum*.

Then being the *Sum* of all the *Squares* which *Constitute* the *Frustum* of a *Square Pyramid*; are to the *Sum* of all the *Circles* which *constitute* the like *Frustum* of a *Right Cone*, in the *Ratio* of 1 : To 0,7854. (Or of 452 : To 355.) Therefore it will be $1 : 0,7854 :: DD + Dd + dd \times \frac{1}{3}H : 0,7854DD + 0,7854Dd + 0,7854dd \times \frac{1}{3}H =$ the *Cone's Frustum*.

That is, in the last *Example*, $1 :: 0,7154 : 1776 : 1394,8704$ the like *Frustum* of a *Right Cone*.

Or, because $\frac{1}{\frac{1}{0,7854}} = 1,273236$ &c. Therefore it may be made $1,273236) DD + Dd + dd \times \frac{1}{3}H (=$ the same *Frustum*. That is, $1,273236) 1776 (1394,87$ &c. As before.

And if you take the *Triple* of this *Divisor*, viz. $1,273236 \times 3$ it will be $3,8197) DD + Dd + dd \times H (=$ the *Frustum*, &c.

Again,

Suppose	1	$x = D - d.$	And $F =$ the <i>Frustum</i> .
Then	2	$DD + Dd + dd = \frac{3F}{H}$	per 7th Step of the Last.
1 $\ominus$ 2	3	$xx = DD - 2Dd + dd.$	
2 $-$ 3	4	$3Dd = \frac{3F}{H} - xx.$	
4 $\div$ 3	5	$Dd = \frac{F}{H} - \frac{1}{3}xx.$	Or $Dd + \frac{1}{3}xx = \frac{F}{H}.$
5 $\times$ H	6	$Dd + \frac{1}{3}xx : \times H = F$	the <i>Frustum</i> $abAB$.

Hence we have another easy *Theorem* for finding the same *Frustum*. T H E-

THEOREM XVI.

To the Rectangle of the Sides of the two Bases, Add one third Part of the Square of their Difference; that Sum being Multiplied into the Height, will produce the Solidity.

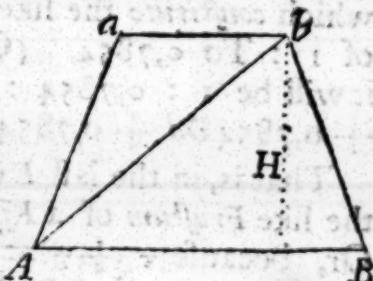
Example. Let $D = 16$. $d = 12$. $H = 9$. as before.
Then $Dd = 192$. $D - d = 4 = x$. $\frac{1}{3}xx = \frac{1}{3} \times 4^2 = 5,3333$.

And $192 + 5,3333 = 197,3333$.

Lastly $197,3333 \times 9 = 1775,9997$ the Solidity of the Frustum of the Square Pyramid. As before.

And $3,81968 \times 1775,9997 = 1394,87$ &c. the like Frustum of a Right Cone. As before.

Either of the two Last Theorems (being rightly applied) will produce the true Solid Content of all Frustums of any kind of Pyramids that are Intercepted between two Parallel and alike Plains or Bases. As above.

But if such Frustums are cut through the Extremities of both Bases by a Diagonal Plain (as Ab in the annexed Figure) into two Parts, Aab , and ABb , called Hoofs: Then the Solidity of those Hoofs, is usually found by dividing the middle Term Dd of the Equation $DD + Dd + dd$ into two Parts, and Adding one of those Parts to the Square of each Base. *A*  *B*

Thus, $DD + \frac{1}{2}Dd : \times \frac{1}{2}H =$ the Great Hoof ABb .

And $dd + \frac{1}{2}Dd : \times \frac{1}{2}H =$ the Lesser Hoof Aab . of the Frustum of any Square Pyramid.

Then $3,8197) DD + \frac{1}{2}Dd : \times H (=$ the Greater Hoof of a Cone.

And $3,8197) dd + \frac{1}{2}Dd : \times H (=$ the Lesser Hoof, &c.

Which are the Theorems proposed by Mr. Dary, in his Book of Gauging Page 36. but do not hold exactly true, for they make the upper Hoof Aab too big, and the lower Hoof ABb as much too little. For indeed Dd should not be Divided into two equal Parts, but rather into two such Parts as have the Ratio of D To d .

Thus, Supposing z the Greater, and x the Lesser Part,
Let it be made $D : d :: z : x$. Ergo, $\frac{dz}{D} = x$. And $\frac{Dx}{d} = z$.
But

But $z + x = Dd$ by *Supposition*. Therefore $Dd - z = x$.
And $Dd - x = z$.

Consequently, $\frac{dz}{D} = Dd - z$. And $\frac{Dx}{d} = Dd - x$.

Whence it follows,

That $\left\{ \begin{array}{l} DD + \frac{DDd}{D+d} : \times \frac{1}{3}H = \text{the Solidity of the Greater Hoof } ABb. \\ dd + \frac{Ddd}{D+d} : \times \frac{1}{3}H = \text{the Solidity of the Lesser Hoof } Aab. \end{array} \right.$
of the *Frustum* of any *Square Pyramid*.

And $\left\{ \begin{array}{l} 3,81968) DD + \frac{DDd}{D+d} : \times H (= \\ 3,81968) dd + \frac{Ddd}{D+d} : \times H (= \end{array} \right\}$ The like *Hoofs* of a *Right Cone*.

Note, In order to avoid many Words in the following Demonstrations, Let $\odot$ signify any Circle in general; and if any two Letters be joined to it, thus, $\odot BA$, &c. it then denotes the Area of such a Circle as those two Letters represent the Radius of.

THEOREM XVII.

The Superficies of every Sphere (or Globe) is Equal to Four times the Area of its greatest Circle.

That is, of a Circle whose Diameter is the Axis of the Sphere.

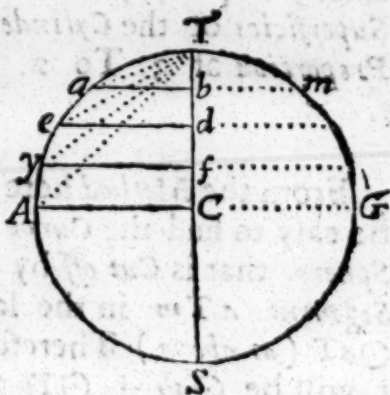
Demonstration.

If any Semi-Circle (as $ATCS$) be Turned or Moved about its Diameter (TS) it will Describe a Solid Body called a Sphere, which will be constituted of an Infinite Series of Concentrick or Parallel Circles, whose Diameters are Chords. *Viz.* $\odot ab$. $\odot ed$. $\odot ef$. &c. per Definition 14.

Consequently, the Superficies of the Sphere, will be composed of the Peripheries of those Circles which constitute its Solidity. Per Definition 20.

Let $D = TS$, the Axis of any Sphere, Then, according to the Property of a Circle, it

will be $\left| \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} D - Tb \times Tb = \square ab. \\ D \times Tb = \square Tb = \square ab. \end{array}$
That is,



There-

Therefore $\left\{ \begin{array}{l} 3 \\ 4 \\ 5 \end{array} \right. \begin{array}{l} D \times Tb = \square aT. \\ D \times Td = \square eT. \\ D \times Tf = \square yT, \text{ \&c.} \end{array}$ For $\square ab + \square Tb = \square aT$.

Hence it's evident, that the Series $\square aT$, $\square eT$, $\square yT$, &c. are in the same Ratio with Tb , Td , Tf , &c. viz. in Arithmetick Progression. Whence it follows, that the $\odot aT =$ the Sum of all the Circle's Peripheries between T and b .

And $\odot eT =$ the Sum of all the Circle's Peripheries between T and d , &c.

Consequently, that the $\odot AT =$ the Sum of all the Circles's Peripheries included between T and C . That is, $\odot AT =$ the Superficies of the Semi-sphere.

And because $\square AC + \square TC = \square AT$, and $\square AC = \square TC$. Therefore $\odot AT = 2\odot AC$ is the Superficies of the Semi-sphere.

Consequently, $4\odot AC$ will be the Superficies of the whole Sphere: Q. E. D.

Example. Suppose the Axis $TS = D = 16$, Then $DD = 256$. And $1 :: 0,7854 : 256 : 201,0624 = \odot AC$. For $\frac{1}{3}D = AC$. Then $201,0624 \times 4 = 804,2496$ the Superficies of the whole Sphere.

Or, because $3,1416$ is Four times $0,7854$. therefore it will always be $1 : 3,1416 :: DD : 3,1416DD$ the Superficies of the Sphere (as before) And it's Equal to the Curve Superficies of a Right Cylinder, whose Diameter and Height are each $= D$ the Axis of the Sphere.

For $3,1416D =$ the Periphery of the Cylinder's Base; and that Multiplied with D its Height, will be $3,1416DD$ the Curve Superficies of the Cylinder, per Theor. 12.

And if to this there be Added the Area of its two Bases (or Ends) Viz. $1,5708DD$. Then it is Evident, that the whole Superficies of the Cylinder will be to that of the Sphere, in the Proportion of 3 To 2.

Scholium.

From the Method here used in proving the last Theorem, it will be easy to find the Curve Superficies of any Segment or Part of a Sphere, that is Cut off by a Right Line or Plane; viz. such as the Segment aTm in the last Scheme, whose Curve Superficies is $\odot aT$ (as above) Therefore (because $\square ab + \square Tb = \square aT$) it will be $\odot ab + \odot Tb =$ the Curve Superficies of that Segment.

But if the Axis TS , and Height Tb of the Segment are given. Then it will be $TS \times Tb = \square aT$, As in the third Step above. Which gives this Proportion or Theorem. Viz.

Viz. $\left\{ \begin{array}{l} \text{As the Axis of the Sphere : Is to the whole Superficies of} \\ \text{the Sphere :: So is the Height of any Segment : To its Curve} \\ \text{Superficies.} \end{array} \right.$

To which, if there be *Added* the *Area* of the *Segment's Base*, the *Sum* will be the *Superficies* of the whole *Segment*.

THEOREM XVIII.

Every Sphere is Equal to two Thirds of its Circumscribing Cylinder.

That is, of a *Cylinder* whose *Height* and *Diameter* of its *Base*, are each *Equal* to the *Axis* of the *Sphere*.

Demonstration.

According to the Work in the last Theorem, it appears, that $\odot ab$. $\odot ed$. $\odot yf$, &c. do constitute the *Solidity* of the *Sphere*; and that $\square aT$, $\square eT$, $\square yT$, &c. are a *Series* of *Terms* in *Arithmetick Progression*, $\square AT$ being the *Greatest Term*, and TC the *Number* of *Terms*. Therefore $\odot AT \times \frac{1}{2}TC =$ the *Sum* of all the *Series*.

per Lemma 2.

And because $\square aT - \square Tb = \square ab$.
 $\square eT - \square Td = \square ed$. $\square yT - \square Tf = \square yd$.
 $\square AT - \square TC = \square AC$, &c. where-
in $\square Tb$, $\square Td$, $\square Tf$, &c. are *Series* of *Squares*, whose *Roots* Tb , Td , Tf , are in *Arithmetick Progression*; $\square TC$ being the *Greatest Term*, and TC the *Number* of *Terms*. Therefore $\odot TC \times \frac{1}{2}TC =$ the *Sum* of all that *Series*, per Lemma 3.

Consequently, $\odot AT \times \frac{1}{2}TC : - \odot TC \times \frac{1}{2}TC =$ the *Sum* of the *Series* $\odot ab$. $\odot ed$. $\odot yf$, &c. which constitute the *Solidity* of the *Half Sphere* ATG : Put $D = 2TC$ the *Axis* of the *Sphere*. Then $\frac{1}{2}D = \frac{1}{2}TC$, and $\frac{1}{4}D = \frac{1}{4}TC$. And because $\square AT = 2\square TC$; Therefore $\odot AT = 2\odot TC = 1,5708DD$. And $1,5708DD \times \frac{1}{4}D = 0,3927DDD$.

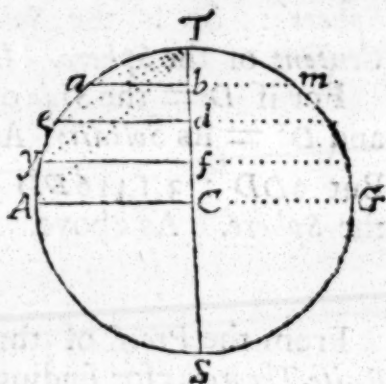
Again $\odot TC \times \frac{1}{2}TC = 0,7854DD \times \frac{1}{2}D = 0,1309DDD$.
Then $0,3927DDD - 0,1309DDD = 0,2618DDD$ the *Solidity* of the *Semi-sphere* ATG .

Consequently, $0,2618DDD \times 2 = 0,5236DDD$ will be the *Solid Content* of the whole *Sphere*, which is *Equal* to $\frac{2}{3}$ of the *Cylinder*, whose *Diameter* of its *Base*, and *Height* $= D$.

For $0,7854DDD =$ the *Solidity* of the *Cylinder*, per Theor. 11. But $\frac{2}{3}$ of $0,7854DDD = 0,5236DDD$, as before. Therefore, &c. As per Theor.

H h h

Exam



Example. Suppose the Axis $D = 16$. Then $DDD = 4096$. And $1 : 0,5236 :: 4096 : 2144,6656$ the Solid Content of that Sphere.

Corollaries.

1. Hence it appears, that the Solid Content of every Sphere, is Equal to its Superficies Multiplied into One Sixth Part of its Axis.

For its Superficies is $3,1416DD$, per Theor. 17.

But $3,1416DD \times \frac{1}{6}D = 0,5236DDD$ the Solid Content, as before

2. And hence it is also Evident, that there is the like Ratio or Habitude between the Cube and its Inscribed Sphere, as is between the Square and its Inscribed Circle. And that is, As the Superficies of any Cube : Is to the Superficies of its Inscribed Sphere :: So is the Solid Content of that Cube : To the Solid Content of the Sphere. (See the Circle's Proportion, Page 401.)

For if $D =$ the Side of the Cube, then $6DD =$ its Superficies, and $D^3 =$ its Solidity. And $3,1416DD =$ the Sphere's Superficies. But $6DD : 3,1416DD :: DDD : 0,5236DDD$ the Solidity of the Sphere. As above.

Scholium.

From the Proof of this Theorem, it will be easie to deduce or Raise Theorems for finding the Solid Content of any Frustum or Segment of a Sphere ; As aTm in the last Figure.

For, we there suppose the Segment aTm to be constituted of an Infinite Series of Circles, which have the same Ratio with all those Circles that constitute the Semi-sphere.

Therefore, it follows, that $\odot aT \times \frac{1}{2}Tb : - \odot bT \times \frac{1}{2}Tb$ will be the Sum of all the Circles intercepted between T and b . Consequently it will be the Solidity of that Segment,

And because $\square ab + \square Tb = \square aT$. Therefore

$\odot ab + \odot Tb \times \frac{1}{2}Tb : - \odot Tb \times \frac{1}{2}Tb =$ the same Solidity.

Let $c = ab$, Half the Segment's Base ; $b = Tb$ its Height ; and $S =$ the Solidity of the Segment or Frustum.

Then $\odot ab = 3,1416cc$. and $\odot Tb = 3,1416bb$.

Consequently, $\frac{3,1416ccb + 3,1416bbb}{2} - \frac{3,1416bbh}{3} = S$.

which being reduced, will become $3ccb + bbb \times 0,5236 = S$.

Or $1,909855)3ccb + bbb (= S$. For $0,5236)1,0000 (1,909855$

which Theorem for finding the Frustum's Solidity:

Note,

Note, Here we suppose the *Height* of the *Segment*, and the *Diameter* of its *Base* to be given: But if the *Axis* of the *Sphere*, and the *Height* of the *Segment* be given; Then putting $D =$ the *Sphere's Axis*, $b =$ the *Segment's Height*, and c as before; it will be $D - b \times b = cc$. Viz. $Dh - hh = cc$.

Therefore $3Dhh - 2hhh = 3cch + hhh$.

Consequ. $3Dhh - 2hhh \times 0,5236 = S$ the *Frustum's Solidity*.

Or $1,90985) 3Dhh - 2hhh (= S$. As before.

Which is a *Second Theorem* for finding the same *Frustum a T m*:

And if it be required to find the *middle Part a m N K*, usually called the *Middle Zone* of a *Sphere*:

Then, because it is supposed that $am = NK$, or which is all one, that $bC = CB$; therefore it is plain, that if *Twice* the *Segment a T m*, be taken from the *Solidity* of the whole *Sphere*, there will Remain the *middle Zone a m N K*.

But because that *Work* is a little *Troublesome*, I shall here shew how to raise a *Theorem* for the doing it.

First, because $AC = yC = eC = aC = TC$. Therefore it will be $\square AC - \square Cf = \square yf$. $\square AC - \square Cd = \square ed$. $\square AC - \square Cb = \square ab$, &c.

Here because $\square AC$. $\square AC$. $\square AC$, &c. are a *Series of Equals*, and Cb the *Number* of all the *Terms*; Therefore $\square AC \times Cb =$ the *Sum* of all that *Series*, per *Lemma 1*.

And $\square Cf$. $\square Cd$. $\square Cb$, &c. being a *Series of Squares* whose *Roots* are in *Arithmetick Progression* beginning at the *Center* or *Point C*, viz. 0 , Cf , Cd , Cb , &c. wherein the *Greatest Term* is $\square Cb$, and *Number* of *Terms* is Cb . Ergo $\square Cb \times \frac{1}{3}Cb =$ the *Sum* of all the *Series*, per *Lemma 3*.

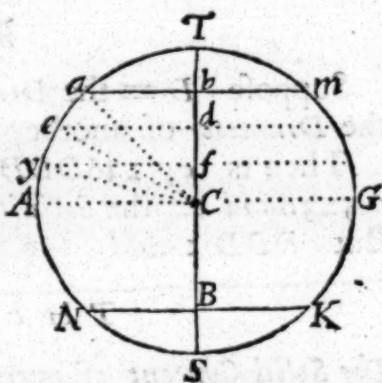
Consequently, the $\odot AC \times Cb : - \odot Cb \times \frac{1}{3}Cb =$ the *Sum* of all the *Series* $\odot yf$. $\odot ed$. $\odot ab$, &c. which do constitute the *Solidity* of the *Half Zone a m A G*.

And because $\square AC - \square Cb = \square ab$. Ergo $\odot AC - \odot ab = \odot Cb$.

Conseq. $\odot AC \times Cb : - \frac{\odot AC + \odot ab \times Cb}{3} = 2\odot AC - \odot ab \times \frac{1}{3}Cb$ will be the *Solidity* of the *Half Zone*.

Put $D = AG = 2AC$. $x = am$. and $H = bB = 2Cb$.

Then $\odot AC = 0,7854DD$. $\odot ab = 0,7854xx$. And if we turn the common *Factor* $0,7854$ into the *Divisor* $1,27323$ and



and then take Triple of that Divisor, viz. 3,8197 (as before in the Frustums of Pyramids) the Result of the precedent Work will produce this following Theorem.

THEOR. XIX. $\left\{ \frac{2DD + xx}{3,8197} : \times H = \right\}$ the middle Zone $amNK$.

THEOREM XX.

Spheres are in Proportion one to another, as the Cubes of their Diameters. (18. e. 12.)

Demonstration.

Suppose D = the Diameter or Axis of any Sphere; and d = the Diameter of another Sphere either Greater, or Lesser.

Then is $0,5236DDD$ = the Solidity of one Sphere, and $0,5236ddd$ = the Solidity of the other Sphere, per Theorem 18. But $DDD : ddd :: 0,5236DDD : 0,5236ddd$. Q. E. D.

THEOREM XXI.

The Solid Content of every Spheroid, is Equal to Two Thirds of its Circumscribing Cylinder.

Demonstration.

Suppose the Figure $NTnSN$ in the Annexed Scheme, to represent a Spheroid, form'd by the Rotation of the Semi-Ellipsis TNS , about its Transverse Axis TS (as per Definition 15)

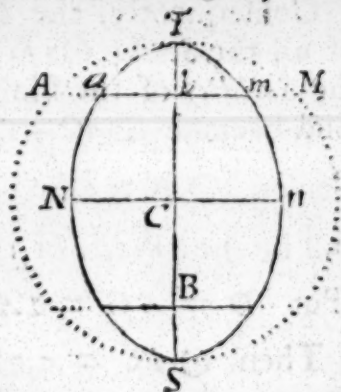
Let $D = TS$ the Length of the Spheroid, and the Axis of its Circumscribing Sphere. And $d = Nn$ the Diameter of the Greatest Circle of the Spheroid.

Then because $\square TC : \square NC :: \square Ab : \square ab$, per Step 3. in Theor. 7. Therefore it will be $DD : dd :: \square Ab : \square ab :: \odot Ab : \odot ab$, &c.

But the Sum of an Infinite Series of such Circles as $\odot Ab$, (whose Diameters are Chords) do constitute the Solidity of the Sphere (as before at Theorem 18.)

And the Sum of an Infinite Series of such Circles as $\odot ab$ (viz. whose Diameters are Ordinates of the Ellipsis) do constitute the Solidity of the Spheroids, per Definition 15.

Ergo $DD : dd :: 0,5236DDDB : 0,5236ddD$ = the Solidity of the Spheroid, per Lemma 6.



But

But $0,5236ddD = \frac{2}{3}$ of the Cylinder, whose Diameter is $= d$, and Height $= D$, per Theorem 11. Q. E. D.

Now from this Proportion between the Sphere and its Inscribed Spheroid, it will be very easie to deduce Theorems for finding the Solid Content, either of the Segment, or Middle Zone of any Spheroid, having the same Height with that of the Sphere.

For $\left\{ \begin{array}{l} \text{As the Solidity of the whole Sphere : Is to the Solidity of the} \\ \text{whole Spheroid} :: \text{So is any Part of the Sphere : To the like} \\ \text{Part of the Spheroid, per Converse to Lemma 6.} \end{array} \right.$

As for Instance; Suppose it were required to find the Middle Zone of any Spheroid.

Let $D = TS$, And $d = Nn$, as above; And $H = bB$. $x = AM$, as in Theorem 19. And let $c = am$.

Then $\left\{ \frac{2DD + xx}{3,8127} \times H = \text{the Middle Zone of the Sphere. And} \right.$

$0,5236D^3 : 0,5236ddD :: \frac{2DD + xx}{3,8127} \times H : \frac{2dd \times H}{3,8197} + \frac{xxdd \times H}{3,8197DD}$
 $= \text{the Middle Zone of the Spheroid.}$

Again, $DD : dd :: xx : cc$. Therefore $\frac{xxdd}{DD} = cc$.

Consequently, $\frac{xxdd}{DD} \times \frac{H}{3,8197} = \frac{cc}{3,8197} \times H$. Which being

taken instead of $\frac{xxdd \times H}{3,8197DD}$ there will arise this following Theorem.

THEOREM XXII. $\left\{ \frac{2dd + cc}{3,8197} : \times H = \right\}$ the middle Zone of the Spheroid, being the very same with Theorem 19.

Note, In the same manner you may Raise Theorems for finding the Segment of a Spheroid, cut off either of its Ends, &c.

THEOREM XXIII.

The Area of every Parabola, is Equal to Two Thirds of its Circumscribing Parallelogram.

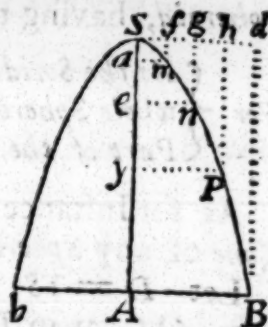
Demonstration.

Let the Figure SAB represent Half a Parabola; make DB Parallel to the Axis SA , and Sd Parallel to the Semi-Ordinate AB . And suppose Sd to be Divided into an Infinite Series

Series of Equidistant Points, As f, g, h , &c. and from those Points imagine a Series of Parallel Lines, viz. fm, gn, hp , &c. to touch the Curve of the Parabola, and meet the Semi-Ordinates ma, ne, yp , &c.

Then according to the Property of the Parabola it will

be {	1	$SA : \square AB :: Sa : \square am$
	2	$SA : \square AB :: Se : \square en$
	3	$SA : \square AB :: Sy : \square yp$, &c.
But		$Sa = fm, Se = gn, Sy = hp. SA = dB$
		Therefore Alternately it will be
3,	4	$\square AB : dB :: \square yp : hp$
2,	5	$\square AB : dB :: \square en : gn$
1,	6	$\square AB : dB :: \square am : fm$, &c.



In these Proportions $\square am, \square en, \square yp$, &c. are a Series of Squares, whose Roots, Sf, Sg, Sh , &c. are in Arithmetick Progression beginning at the Point S . And because the Lines bp, gn, fm , &c. have the same Ratio; Therefore they are as such a Series of Squares, wherein dB is the Greatest Term, and Sd the Number of Terms.

Consequently, $\frac{dB \times Sd}{3} =$ the Sum of all those Lines, per Lem. 3.

But $SA \times AB = dB \times Sd$. Therefore $\frac{SA \times AB}{3} =$ the Sum of all that Series of Lines; but all those Lines do constitute the Area of the Semi-Parabola's Complement: viz. the Area of what Half the Parabola SAB , wants of completing or filling up the Parallelogram $SdAB$.

Wherefore $SA \times AB : -\frac{1}{3} SA \times AB = \frac{2SA \times AB}{3}$ will be the Area of Half the Parabola SAB .

Consequently, $\frac{2}{3} SA \times bB$, will be the Area of the whole Parabola bSB . Q. E. D.

Example. Suppose the Base or Greatest Ordinate of a Parabola, to be $bB = 24$. and its Intercepted Diameter or Axis be $SA = 33$. Then $2SA \times bB = 66 \times 24 = 1584$. And $3) 1584$ (528 the Area of that Parabola.

THEOREM XXIV.

Every Parabolick Conoid, is Equal to One Half of its Circumscribing Cylinder.

Demonstra-

Demonstration.

If any *Semi-Parabola* (as *BSA*) be turned or mov'd about its *Axis* (*SA*) it will form a *Solid Parabolick-Conoid*, constituted of an *Infinite Series of Circles*, viz. $\odot ba$, $\odot fe$, $\odot gy$, &c. per *Defin.* 17.

Now according to the *Property* of every *Parabola*, it will be, $SA : AB :: AB : \frac{\square AB}{SA} = L$, the *Latus Rectum*.

$$\text{Then } \begin{cases} Sa \times L = \square ba \\ Se \times L = \square fe \\ Sy \times L = \square gy, \text{ \&c.} \end{cases}$$

Here $Sa \times L$, $Se \times L$, $Sy \times L$, &c. are a *Series of Terms* in *Arithmetick Progression*. Therefore $\square ba$, $\square fe$, $\square gy$, &c. are also a *Series of Terms* in the same *Progression*, beginning at the *Point S*, wherein $\square AB$ is the *Greatest Term*; And SA the *Number of all the Terms*. Therefore $\square AB \times \frac{1}{2}SA =$ the *Sum* of all the *Series*, per *Lemma 2*.

Consequently, $\odot AB \times \frac{1}{2}SA =$ the *Sum* of all the *Series* of $\odot ba$, $\odot fe$, $\odot gy$, &c. which do constitute the *Solidity* of the *Conoid*.

And putting $D = 2AB$, and $H = SA$;

Then $0,7854DD \times \frac{1}{2}H = 0,3927DDH$ will be the *Solid Content* of the *Conoid*; which is just *Half* the *Cylinder* whose *Base* $= D$, and *Height* $= H$. See *Theorem 11*. Q. E. D.

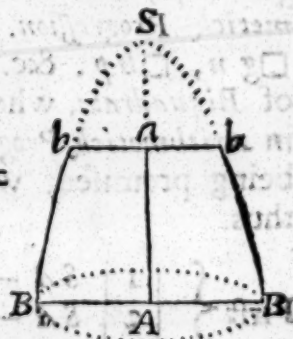
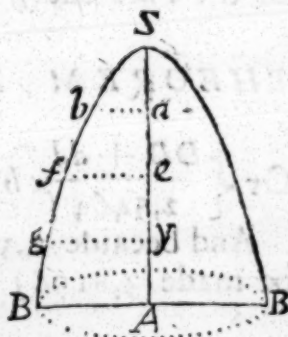
This being understood, it will be *Easie* to *Raise a Theorem* for finding the *Lower Frustum* of any *Parabolick Conoid*.

For supposing $b = aA$ the *Height* of the *Frustum*, and $p = Sa$ the *Height* of the Part bSb cut off. Then $b + p = SA$ the *Height* of the whole *Conoid*.

Consequently $\frac{\odot AB \times b + \odot AB \times p}{2} =$ the *Solidity* of the whole *Conoid*.

And $\frac{\odot ba \times p}{2} =$ the *Solidity* of the Part cut off.

$$\begin{array}{l|l} \text{Ergo} & 1 \left\{ \frac{\odot AB \times b + \odot AB \times p - \odot ba \times p}{2} = \right. \\ & \quad \text{the Solidity of the Frustum.} \\ \text{But} & 2 \quad b + p : \square AB :: p : \square ba \\ \text{Conseq.} & 3 \quad b + p : \odot AB :: p : \odot ba \\ & 4 \quad \odot AB \times p = \odot ba \times b + \odot baxp \end{array}$$



$$\begin{array}{r|l}
 4 - \odot ba \times p & 5 \quad \odot AB \times p : - \odot ba \times p = \odot ba \times b \\
 1 \times \frac{1}{2} & 6 \quad \odot AB \times b : - \odot AB \times p : - \odot ba \times p = 2F \\
 6 - 5 & 7 \quad \odot AB \times b = 2F : - \odot ba \times b \\
 7 + \odot ba \times b & 8 \quad \odot AB \times b : - \odot ba \times b = 2F \\
 8 \div \frac{1}{2} & 9 \quad \frac{\odot AB + \odot ba}{2} : \times b = F \text{ the Frustum's Soli-} \\
 & \text{(dity).}
 \end{array}$$

Let $D = 2AB$, as before; And $d = 2ba$ the Diameter of the Part cut off. Then we shall have this following

THEOREM XXV. $\{ 0,3927DD + 0,3927dd : \times b = \text{the Solidity of the Frustum required.}$

Or $\left\{ \frac{DD + dd}{2,5464} \times b = \text{the Frustum. For } ,3927 \right\} 1,0000 (= 2,5464$

And because $2,5464 + \frac{2,5464}{1} = 3,8196$. Therefore it may be made $3,8196) DD + dd : \times 1\frac{1}{2}b (= \text{the same Frustum, \&c.}$

Note, The Reason why I have reduced this Theorem to have the same Divisor with those at the Frustums of Pyramids &c. will best appear further on, viz. when they all come to be applied to Practice in Gauging.

THEOREM XXVI.

Every Parabolick Spindle (or Pyramidoid) is Equal to Eight Fifteenths of its Circumscribing Cylinder.

Demonstration.

If any Acute Parabola, as bSB , be Turned or Moved, about its Greatest Ordinate bAB , it will form a Solid called a Parabolick Spindle, constituted of an Infinite Series of $\odot ma$, $\odot ne$, $\odot py$, &c. per Definition 18.

Let us suppose the Line Sd Parallel to AB , &c. As at Theor. 23. Then it hath already been proved, that the Lines fm , gn , bp , &c. are a Series of Squares whose Roots are in Arithmetick Progression. Consequently, their Squares, viz. $\square fm$, $\square gn$, $\square bp$, &c. will be a Series of Biquadrats, whose Roots will be in Arithmetick Progression; which being premised, we may proceed thus.

$$\text{First } \left\{ \begin{array}{l} 1 \quad SA - fm = ma \\ 2 \quad SA - gn = ne \\ 3 \quad SA - bp = py \text{ \&c.} \end{array} \right.$$



$$\begin{array}{l|l|l} 1 \odot 2 & 4 & \square SA - 2SA \times fm + \square fm = \square ma \\ 2 \odot 2 & 5 & \square SA - 2SA \times gn + \square gn = \square ne \\ 3 \odot 2 & 6 & \square SA - 2SA \times bp + \square bp = \square py, \&c. \end{array}$$

1. In these *Equations* the $\square SA$, $\square SA$, $\square SA$ being a *Series* of *Equals*, and AB the *Number* of all the *Terms*; Therefore it will be $\square SA \times AB =$ the *Sum* of all the *Series*, per *Lemma* 1.

2. Because fm , gn , bp , &c. are as a *Series* of *Squares* wherein SA is the *Greatest Term*, and AB the *Number* of all the *Terms*. Therefore $\frac{2SA \times SA \times AB}{3} = \frac{2\square SA \times AB}{3}$ will be the *Sum* of all that *Series*, per *Lemma* 3.

3. And the $\square fm$, $\square gn$, $\square bp$, &c. will be a *Series* of *Terms* in the *Ratio* of *Biquadrates*, as above, $\square dB = \square SA$ being the *Greatest Term*, and AB the *Number* of all the *Terms*; Therefore it will be $\frac{\square SA \times AB}{5} =$ the *Sum* of all the *Series*, per *Lemma* 5.

Whence it follows, that $\square SA \times AB - \frac{2\square SA \times AB}{3} + \frac{\square SA \times AB}{5} =$ the *Sum* of all the *Series* of $\square ma$, $\square ne$, $\square py$, &c.

That is, $\frac{8\square SA \times AB}{15} =$ the *Sum* of all the *Series* of $\square ma$, $\square ne$, $\square bp$, $\square dB$, &c.

Consequently, $\frac{8\odot SA \times AB}{15} =$ the *Sum* of all the *Series* of $\odot ma$, $\odot ne$, $\odot py$, &c. which do constitute the *Solidity* of Half the *Spindle*, viz. of SAB .

Therefore, putting $D = 2SA$, and $H = 2AB$ (viz. bAB) it will be $0,41888DDH =$ the *Solidity* of the whole *Parabolick Spindle* bSB , being $\frac{8}{15}$ of $0,7854DDH$ the *Solidity* of its *Circumscribing Cylinder*. Q. E. D.

From hence we may also *Raise* a *Theorem* for finding the *Fru-
stum* $SApy$ of the last *Figure*.

For $\odot SA$ being the *Greatest Term*; $\odot py$ the *Least Term*, and Ay the *Number* of all the *Terms* or *Circles* included between A and y .

$$\begin{array}{l|l} \text{Therefore} & 1 \left\{ \begin{array}{l} \square SA - \frac{2SA \times bp}{3} + \frac{\square bp}{5} : \times Ay = z \text{ the Sum} \\ \text{of all the Series } \square SA, \square ma, \square gn, \square py \end{array} \right. \\ & 2 \left\{ \begin{array}{l} 3\square SA - 2SA \times bp + \frac{3\square bp}{5} : \times Ay = 3z \end{array} \right. \end{array}$$

I i i 2 ÷

$$\begin{array}{ll}
 2 \div Ay & 3 \quad 3\Box SA - 2SA \times bp + \frac{3\Box bp}{5} = \frac{3z}{Ay} \\
 \text{But} & 4 \quad \Box SA - 2SA \times bp = \Box py - \Box bp. \text{ Per 6th Step.} \\
 3 - 4 & 5 \quad 2\Box SA + \frac{3\Box bp}{5} = \frac{3z}{Ay} - \Box py + \Box bp \\
 5 + \&c. & 6 \quad 2\Box SA + \Box py - \frac{2}{5}\Box bp = \frac{3z}{Ay} \\
 \text{Consequ.} & 7 \quad 2\odot SA + \odot py - \frac{2}{5}\odot bp : \times \frac{1}{3}Ay = z. \text{ the}
 \end{array}$$

Sum of all the Series of $\odot SA$. $\odot ma$. $\odot ne$. $\odot py$, which do constitute the Solidity of the Frustum $SApy$. Therefore putting $D = 2SA$, As before, $C = 2py$, $x = 2bp$, And $H = Ay$ it will be $1,5708DD + 0,7854CC - 0,31416xx : \times \frac{1}{3}H =$ the Frustum $SApy$. And if we make $L = 2H$.

Then $1,5708DD + 0,7854CC - 0,31416xx : \times \frac{1}{3}L =$ the Double of that Frustum, being the Middle Zone. And by turning these Factors into one common Divisor, as in the Frustum of the Conoid at Theorem 25. Page 424. there will arise this following Theorem.

THEOR. XXVII. $\{ 3,8196) 2DD + CC - 0,4xx : \times L (=$
the Middle Zone of a Parabolick Spindle.

It may be here expected that I should now proceed to shew how the Area of any Hyperbola, and the Contents of such Solids as may be form'd by the Rotation of that Figure about its Axis, &c. may be found: But because those things cannot be exactly perform'd, by any certain or settled Theorems; As these of the Circle, Ellipsis, and Parabola have been: I've therefore omitted them, and refer the Reader to Dr. Wallis's Algebra, Chap. 90. &c. Or to Philosoph. Transf. Number 34. wherein he may find the Method of forming Infinite Series relating to the Squaring of an Hyperbola, &c. which are too tedious to be fully Explain'd and Demonstrated in this small Treatise, it being only intended as an Introduction, the which I shall here conclude.

F I N I S.

AN APPENDIX OF

Practical Gauging.

THE Art of Gauging is that Branch of the *Mathematicks* called *Stereometry*, or the *Measuring of Solids*; because the *Capacities* or *Contents* of all sorts of *Vessels* used for *Liquors*, &c. are computed as tho' they were really *Solid Bodies*; which any one that hath made himself Master of the foregoing Parts of this *Treatise*, may easily understand, without any further Directions.

However, because it is not to be supposed, that every one which designs to undertake the Office or Employment of a Gauger, hath made so great a Progress in *Mathematical Learning*; I have therefore presented the young Gauger with this Appendix, wherein I've only Inserted such Rules as are useful in Gauging, and have been already *Demonstrated* in this *Treatise*. But herein, I pre-suppose, that he hath acquired, (or if not, it is very requisite he should acquire) a competent Knowledge both in *Arithmetick* and *Geometry*. That is,

I. In *Arithmetick* he should understand the *Principal Rules* very well; especially *Multiplication* and *Division*, both in whole Numbers and *Decimal Parts*; (which may be easily Learned out of the 2d, 3d, and 5th Chapters of Part 1.) That so he may be ready in computing the *Contents* of any *Vessel*, and casting up his *Gauges* by the Pen only, viz. without the help of those *Lines* of Numbers upon *sliding Rules*, so much Applauded, and but too much Practised, which at best do but help to guess at the Truth, and may justly be called an *Idle Ignorant Way* of doing Business, if compared with that of the Pen.

II. In *Geometry* the Gauger should understand not only how to take *Dimensions* (which is best Learned by Practice) but also how to Divide any *Irregular Figure* or *Superficies*, As *Brewer's Backs*, or *Coolers*, &c. into the easiest and fewest *Regular Figures* they

will admit of; that so their *Area's* may be truly computed with the least Trouble. And this may be Learned (*with a little Care and Diligence*) out of the 1st 2d and 5th Chapters of Part 3. which the Gauger should be well acquainted with.

Also he ought to have so much Skill in Solids, as to be able, even at sight (*but this must be acquired by Experience*) to determine what sort of Figure any Vessel is of, (*viz. any Tun, or Close Cask*) or what Figure it may be best reduced to; so that its Dimensions may be truly taken, and the Content thereof computed with the least Error. I say with the least Error, because it is very Difficult, if not Impossible, to do it Exactly; for there is not any Tun or Cask, &c. so Regularly made, as by the Rules of Art it is required to be.

III. Besides the aforementioned, the young Gauger must know, that all Dimensions useful in Gauging, are to be taken in Inches, and Decimal Parts of an Inch; and if they are taken in any other Measures, as Feet, or Yards, &c. those Measures must be Reduced to Inches (*See Sect. 4. Page 42.*) because the Contents of all sorts of Vessels (*taken Notice of in Gauging*) are computed by the Standard Gallon of its kind, whose Content is known to be a certain Number of Cubick Inches. That is, the Beer or Ale Gallon contains 282. the Wine Gallon 231. and the Corn Gallon 268,8 Cubick Inches. (*See the five Tables, &c. in Page 34, 35, 36. which I here suppose the Gauger to have learned perfectly by Heart*) Consequently, if either the Superficial, Or Solid Content of any Vessel, as Back, Tun, Cask, &c. be once computed in Cubick Inches, it will be easie to know how many Gallons, either of Ale, Wine, Or Corn, that Vessel will hold.

Note, I have here said the Superficial Content in Cubick Inches, which may seem to be very Improper, according to the Definition given of a Superficies in Page 279. But you must know, that in the Business of Gauging, all Superficies or Area's are always understood to be one Inch deep; otherwise it could not be said. (*As in the Gauger's Language it is*) that the Area of such a Back, Or of such a Circle, &c. is so many Gallons.

These things being very well understood, the young Gauger will be fitly prepared to understand the following Problems, which are such as have (*most of them*) been already proposed, in the foregoing Parts of this Treatise; and are here applied to Practice; And therefore I shall for Brevities sake, often refer to those Theorems and Problems.

Section 1. To find the Area of any Right-lin'd Superficies in Gallons.

Problem 1.

To find the Area of any Square Tun, Back or Cooler, &c. either in Ale, Wine, or Corn Gallons.

Rule { Multiply the given Length, or Breadth (being here Equal) into it self, and the Product will be the Area in Inches;
Then Divide that Area by 282. Or 231. Or 268,8. and the Quotient will be the Area required.

Example. Suppose the Side of a Square Tun, Back or Cooler be 124,5 Inches; What will its Area be in Gallons?

First $124,5 \times 124,5 = 15500,25$ the Area in Inches.

Then 282) 15500,25 (54,96 Ec. the Area in Ale Gallons.

And 231) 15500,25 (67,10 Ec. the Area in Wine Gallons.

Or 268,8) 15500,25 (57,66 Ec. the Area in Corn Gallons.

But if any one would rather work by Multiplication than Division, he may Turn or Change any Divisor into a Multiplier, if he Divide Unity or 1. by that Divisor. (vide Problem 3. Page 402.)

Thus 282) 1,000000 (0,003546 the Multiplier for Ale Gallons.

And 231) 1,000000 (0,004329 the Multiplier for W. Gallons.

Or 268,8) 1,000000 (0,003722 the Multiplier for C. Gallons.

Consequently $15500,25 \times 0,003546 = 54,96$ Ec. the Area in Ale Gallons as before; And so on for the rest.

Problem 2.

To find the Area of any Tun, Back or Cooler, in the form of a Right-angled Parallelogram, in Ale Gallons, &c.

See the Rule for finding its Area in Inches, at Prob. 1. Page 333
Then either Divide (or Multiply) that Area as above, and you will have the Area in Gallons.

Example. Suppose the Length of a Brewer's Tun, Back or Cooler be 217,5 Inches, and its Breadth 85,6 Inches; What will its Area be in Ale or Beer Gallons, &c.

First $217,5 \times 85,6 = 18648$. Then 282) 18648 (66,12 Ec.

Or $18648 \times 0,003546 = 66,12$ Ec. the Area required. Ec.

Problem 3.

To find the Area of any Triangular Tun, Back or Cooler, in Ale Gallons, &c.

See the Rule for finding its Area in Inches, at Prob. 3. Page 334. Then Divide (or Multiply) that Area as before, and you will have the Area required.

Example. If the Length of the Base of a Triangular Cooler be 86,4 Inches; and its Perpendicular Breadth be 57 Inches; What will its Area be in Ale Gallons?

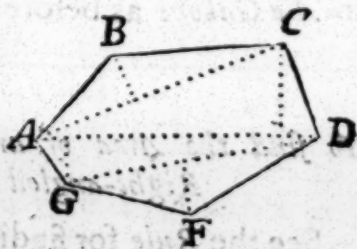
First $86,4 \times \frac{1}{2} = 2462,4$. Then $282 \mid 2462,4$ (8,73 &c.

Or $2462,4 \times 0,003546 = 8,73$ &c. the Area in Ale Gallons.

Proceeding thus, you may easily find the Area of any Tun, Back or Cooler, whether it be in the Form of a Rhombus, Rhomboides, Trapezium, or of any other Polygon, either Regular, or Irregular, in Ale or Beer Gallons, &c. if you first Divide it into Triangles, and then find the Area's of those Triangles; (As in the 2d, 4th, 5th and 6th Problems in Chapter 5. Part 3.) The Sum of those Area's being Divided (or Multiplied) by its proper Divisor (or Multiplier) as above, will give the Area required.

Now the Practical Way of Dividing any Polygonious Tun, Back, &c. into Triangles, is by Help of a chalk'd Line, such as the Carpenters use; and may be thus performed.

Suppose any Brewer's Tun, Back or Cooler, in the Form of the Annexed Figure *ABCD FG*. Let one End of the chalk'd Line be fastened with a Nail (or otherwise) in any Corner or Angle of the Back, as at *A*; Then straining it to the Angle at *C*, strike the Diagonal Line *AC*, upon the Bottom of the Back; And straining it again to the Angle *D*, strike another Diagonal Line, as *AD*; And so on for the Diagonal Line *GD*, &c. Then having Mark'd out all the Diagonals, the Perpendiculars may be thus found; fasten (as before) one End of the chalk'd Line in the Angle *B*, and then by moving it to and fro upon the stretch, find out the nearest Distance between the Angle at *B*, and the Diagonal Line *AC*, there strike a Line, and it will mark out the Perpendicular from *B* to the Line *AC*; and so on for the other Perpendiculars, which being all mark'd out upon the Bottom of the Back, Measure them, and each Diagonal by a Line of



Inches, &c. And then the Area of that Back may be computed as directed above.

And here by the way it may be observed; That the Number of Triangles will always be Less by Two; and the Number of the Diagonal Less by Three, than the Number of the Sides of any Right-lined Figure that is so Divided.

Having found (as above) the true Area of any Brewer's Back or Cooler, (which according to the Laws of Excise, ought always to be fix'd or immoveable) the next thing will be, to find out the true Dipping or Gauging Place in that Back, that so the true Quantity of Worts may be computed or cast up at any Depth; which may be thus done.

1. When the Bottom of the Back is covered all over, (of any Depth) either with Worts or Liquor (viz. Water) then Dip it in Eight or Ten several Places (more or less according to the Largeness of the Back) as remote and equally distant one from another as you well can, noting down the Wet Inches and Decimal Parts of every Dip.

2. Divide the Sum of all those Dips or Wet Inches by the Number of Places you Dip'd in, and the Quotient will be the Mean Wet of all those Dips.

3. Lastly, find out such a Place by the Side of the Back (if you can) that just Wets the same with that Mean Dip, and make a Notch or Mark there for the true and constant Dipping Place of that Back. Then if any Quantity of Worts (which do cover the whole Back) be Dipped or Gauged at that Place, and the wet Inches so taken be Multiplied into the Area of the Back in Gallons, the Product will shew what Quantity (viz. how many Gallons) of Worts are in that Back at that Time, provided the Sides of the Back do stand at Right-angles with its Bottom.

Section 2. To find the Area of any Circular and Elliptical Superficies in Gallons.

1. I have Demonstrated in Chapter 6. Part 3. And Theorem 3, 5, 6. Part 5. that the Periphery of the Circle whose Diameter is Unity or 1. is 3,14159265 &c. (or for common Use 3,1416) And that its Area is 0,78539816 &c. (Or 0,7854 fere.)

2. Also, that the Peripheries of all Circles are in Proportion one to another, as their Diameters are; And their Area's are in Proportion to the Squares of their Diameters. That is,

As 1 : 3,1416 :: the Diameter of any Circle : To its Periphery.
And 1 : 0,7854 :: the Square of the Diameter : To the Area.

Upon

Upon these Two Proportions depends the Solution of all the Common or Practical Questions about a Circle. See Page 402. and 403.

Problem 4.

The Diameter of any Circle being given in Inches; To find the Periphery.

Rule { Multiply the given Diameter with 3,1416. and the Product will be the Periphery required. See Prob. 1. Page 402.

Example, Suppose the Diameter of a Circle be 54,5 Inches, and it were required to find its Periphery.

Then $54,5 \times 3,1416 = 171,21$ &c. Inches, is the Periphery required.

The Converse of this is Easie, viz. by having the Periphery given; To find the Diameter. See Prob. 3. Page 402.

Problem 5.

The Diameter of any Circle being given (in Inches) To find its Area in Gallons.

Rule { Multiply the Square of the proposed Diameter into 0,7854. and the Product will be the Area in Inches (vide Prob. 2. Page 402.) That Area being Divided by 282 or 231 &c. the Quotient will be the Area required.

Example. Suppose the given Diameter be 54,5 Inches, As above. First $54,5 \times 54,5 = 2970,25$. And $2970,25 \times 0,7854 = 2332,83$ the Area in Inches.

Then $282 \mid 2332,83$ (8,2724 the Area in Ale or Beer Gallons.

And $231 \mid 2332,83$ (10,0988 the Area in Wine Gallons.

Or $268,8 \mid 2332,83$ (8,6788 the Area in Corn Gallons.

But these Area's in Gallons may be much Easier found, without knowing the Circle's Area in Inches as above; by having the Square of the Diameter of that Circle whose Area is one Gallon; which may be thus found, per Theorem 6. Page 401.

$0,785398 : 1 :: 282 : 359,05$ the Square of the Diameter of the Circle whose Area is 282 Cubick Inches, viz. one Ale Gallon.

And from this Proportion will arise these following Divisors.

Viz. $0,785398 \mid 282,000000$ (359,05 will be a Divisor for A. G.

And $0,785398 \mid 231,000000$ (294,12 will be a Divisor for W. G.

Or $0,785398 \mid 368,800000$ (342,24 will be a Divisor for C. G.

If the *Square* of the *Diameter* of any *Circle*, be *Divided* by any one of these *constant* or *fix'd Divisors*, the *Quotient* will shew that *Circle's Area* in their respective *Gallons*. As for Instance in the *Last Circle*, whose *Square* of its *Diameter* is 2970,25.

Then 359,05) 2970,25 (8,2725 the *Area* in *A.G.*
 And 294,12) 2970,25 (10,0988 the *Area* in *W.G.* } As before.
 Or 342,24) 2970,25 (8,6788 the *Area* in *C.G.*

Now these *Divisors* may be turned into *Multiplicators*, by *Dividing Unity* or 1. As in *Page 429*. Or rather by *Dividing the Area* in *Inches* of that *Circle* whose *Diameter* is 1.

That is, 0,785398 by 282. Or by 231. &c.

Thus 282) 0,785398 (0,002785 the *Multiplicator* for *Ale Gal.*

And 231) 0,785398 (0,003399 the *Multiplicator* for *Wine Gal.*

Or 268,8) 0,785398 (0,002922 the *Multiplicator* for *Corn Gal.*

These *Multiplicators* are the *respective Area's* of a *Circle* whose *Diameter* is 1. And therefore, if the *Square* of the *Diameter* of any *Circle*, be *Multiplied* with any of these *Numbers*, the *Product* will be that *Circle's Area* in *Gallons* of the same *Name*.

Viz. 2970,25 $\times$ 0,002785 = 8,2725 the *Area* in *A.G.* as above.

And 2970,25 $\times$ 0,003399 = 10,0988 the *Area* in *W. Gal.* &c.

Thus you see, that if the *Diameter* of any *Circle* be given in *Inches*, there are three several *Ways* of finding its *Area* in *Gallons*, and all equally *True*; But that which is perform'd by the *constant Divisors* is most generally *Practised*.

Problem 6.

The *Transverse* (or *Longest Diameter*) and the *Conjugate* (or *shortest Diameter*) of any *Elliptical Superficies* being given;
 To find its *Area* in *Gallons*.

Rule { Multiply the Two *Diameters* (*viz.* the *Length* and *Breadth*) together, and Divide their *Product* by 359,05 for *Ale Gallons*, or 294,12 for *Wine Gallons*, &c. The *Quotient* will be the *Area* required. See *Theorem 7. Page 406.*

Example, Suppose the *Longest Diameter* to be 73,5 *Inches*, and the *Shortest Diameter* to be 51,6 *Inches*; What will the *Area* be in *Ale Gallons*?

First 73,5 $\times$ 51,6 = 3792,6. Then 359,05) 3792,6 (10,56 the *Area* in *Ale Gallons*. Or 294,12) 3792,6 (12,89 the *Area* in *Wine Gallons*, &c.

K k k

Note,

Note, The Two last Problems are of great Use in Gauging of Worts amongst Country Victulars, who generally Brew but short Lengths of Ale (viz. perhaps between 20 and 60 Gallons at a Brewing) and Cool their Worts in several small Open Vessels or Tubs, whose Bases or Bottoms are either a Circle, or an Ellipsis, having their Sides but Low, and are most commonly Wider at the Top than at the Bottom.

Now a Practical Way of computing the Quantity of Worts that are at any Time in one of those Open Tubs, is briefly thus: When the Tub is Dry, find the true Area of its Bottom, according to its Figure (as above) And either mark that Area on the out Side of the Tub (*which was the way I generally used to order, because the Victulars did often lend their Cooling Tubs one to another*) Or else Number the Tub, and Enter its Area (and its Number) into the Stock Book; Then, when any of those Tubs hath Worts in it, take the Diameter of the Surface or Top of the Worts, and find that Area, Adding it and the Bottom Area together; If either the Half Sum of those Two Area's, be Multiplied with the Depth of the Worts; (*taken as near the Middle of the Tub as you well can*) Or, if the Sum of those Two Area's, be Multiplied with Half the Depth (*so taken*) the Product will shew the Quantity of those Worts very near the Truth.

Problem 7.

The Diameter of any Circle, and the Versed Sine (viz. the Height) of any Segment being given; To find the Area of that Segment in Gallons.

In the 404 and 406 Pages, you have Two Ways (*and their Examples*) of finding the Area of any Segment of a Circle in Inches; Then if that Area in Inches be Divided by 282. or 231 &c. the Quotient will be its Area in Gallons. But because the Area of any such Segment may be readily found in Gallons (*without finding its Area in Inches*) by help of a Table of Segments, whose Construction is laid down in the Problem, Page 405 &c. I have here Inserted a Compendium of such a Table, which will serve very well for Common Practice, not only to find the Area of any Segment of a Circle in Gallons; but also to find the Number of Gallons that are either Drawn out, or Remaining in any Cylinrick Vessel lying along: Or if any Close Cask (*being first Reduced to a Cylinder*) its Axis Lying Parallel to the Horizon; usually called the Ullage of a Cask; As shall be shewed further on.

*A Table of the Segments of a Circle, whose Area is Unity or 1.
The Diameter being Divided by Parallel Chord Lines into
100 Equal Parts.*

V. S. Segment	V. S. Segment	V. S. Segment	V. S. Segment
1 0,0017	26 0,2066	51 0,5127	76 0,8155
2 0,0048	27 0,2178	52 0,5255	77 0,8262
3 0,0087	28 0,2292	53 0,5382	78 0,8369
4 0,0134	29 0,2407	54 0,5509	79 0,8474
5 0,0187	30 0,2523	55 0,5635	80 0,8576
6 0,0245	31 0,2640	56 0,5762	81 0,8677
7 0,0308	32 0,2759	57 0,5888	82 0,8776
8 0,0375	33 0,2878	58 0,6014	83 0,8873
9 0,0446	34 0,2998	59 0,6140	84 0,8968
10 0,0520	35 0,3119	60 0,6265	85 0,9059
11 0,0598	36 0,3241	61 0,6389	86 0,9149
12 0,0680	37 0,3364	62 0,6514	87 0,9236
13 0,0764	38 0,3486	63 0,6636	88 0,9320
14 0,0851	39 0,3611	64 0,6759	89 0,9402
15 0,0941	40 0,3735	65 0,6881	90 0,9480
16 0,1032	41 0,3860	66 0,7002	91 0,9554
17 0,1127	42 0,3986	67 0,7122	92 0,9625
18 0,1224	43 0,4112	68 0,7241	93 0,9692
19 0,1323	44 0,4238	69 0,7360	94 0,9755
20 0,1424	45 0,4365	70 0,7477	95 0,9813
21 0,1526	46 0,4491	71 0,7593	96 0,9866
22 0,1631	47 0,4618	72 0,7708	97 0,9913
23 0,1738	48 0,4745	73 0,7822	98 0,9952
24 0,1845	49 0,4873	74 0,7934	99 0,9983
25 0,1955	50 0,5000	75 0,8045	100 1,0000

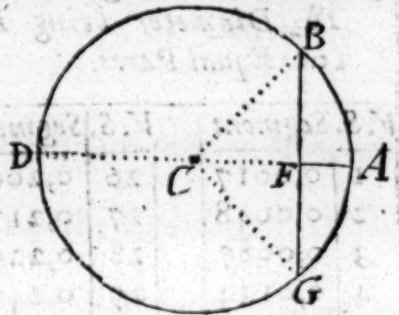
The Use of this Table of Segments depends upon the following Proportion.

Viz. $\left\{ \begin{array}{l} \text{As the Diameter of any proposed Circle : Is to 100 (the} \\ \text{Diameter of the Tabular Circle) :: So is the Height of any} \\ \text{Segment of the proposed Circle : To a Versed Sine in the Table,} \end{array} \right.$

Then if the Tabular Segment, which stands against that Versed Sine, be Multiplied into the Circle's Area (either in Inches or Gallons) the Product will be the Area of the Segment required (of the same Name) viz. If the Circle's Area be Inches, the Segment will be Inches : If Gallons, the Segment will be Gallons.

Example. Let the Diameter of the given Circle be $DA = 62,5$ Inches; And the Height of the Segment sought be $FA = 20$ Inches; What will its Area be in Ale Gallons?

First, the Area of the whole Circle will be 10,8793 Ale Gallons. Per Problem 5. And the Proportion will stand thus, $62,5 : 100 :: 20 : 32$ the Versed Sine of the Table, whose Segment is 0,2759



Then $10,8793 \times 0,2759 = 3,0016$ Ale Gallons, being the Area of the Segment BAFG, as was required. The like may be done for Wine Gallons, Corn Gallons, Or Inches.

And upon Occasion, the like Segments of any Ellipsis may be easily found. See the Proportions in the Corollaries to the 7th and 8th Theorems Page 406 &c. to which I here for Brevity's sake refer the Reader.

Section 3. To compute the Contents of such Vessels (*viz.* Tuns, &c.) as are in the Form of the following Solids.

Note, Before the young Gauger proceed to these Computations, he should be well acquainted with such Solids as are Defin'd in Page 396 and 397. And then he will easily understand what sort of Figures are meant in the following Problems, without the Repetition of many Words.

Problem 8.

To find the Content of any Prism, whose Sides are Parallelograms; what Form soever its Base is of.

That is, To compute the Content (in Gallons) of any Tun, &c. whose Sides are Parallelograms which stand Upright, or at Right Angles with its Bottom.

First find its Solid Content in Inches by Theorem 9. Page 408. Then Divide that Content by 282. Or 231. Or by 268,8 the Quotient will shew the Content in their respective Gallons, *viz.* in Ale, Wine, or Corn Gallons.

Or elle Multiply the Content in Inches with 0,003546 Or 0,004329 &c. (See the Multipliers Page 429.) those Products will be the Content in their respective Gallons.

Or otherwise thus.

Find the true Area of the Tun's Base or Bottom, as directed in Section 1. Page 429. That Area being Multiplied with the Tun's Height (*viz.* Depth within) will produce the Content in Gallons, As before.

I take

I take the Work of this Problem to be so very Easie it needs no Example.

Problem 9.

To find the Content of any Pyramid (in Gallons) whose Base is bounded with Right Lines.

Every Pyramid is One Third Part of its Circumscribing Prism, per Theorem 10. Page 409. Therefore

If the Area of the Base of any Pyramid in Gallons, be Multiplied into One Third of its Perpendicular Height; Or if One Third of that Area be Multiplied with the whole Height, either of those Products will be the Content of the Pyramid in Gallons, &c.

But the Content of any Square Pyramid, may be easily found in Gallons by this Rule.

Rule { Square the Side of its Base; and Multiply that Square with the Perpendicular Height; Then Divide that Product by 846 = 282×3 for Ale Gallons; Or by 693 = 231×3 for W. Gallons; Or by 806,4 = $268,8 \times 3$ for Corn Gallons; The Quotient will be the Content required.

Or if you Multiply the said Product with 0,001182 for A. G. or with 0,001443 for W. G. or Lastly with 0,001241 for C. G. the Result will be the Content required. As before.

Problem 10.

To find the Content (in Gallons) of the Frustum of any Square Pyramid, cut off by a Plain Parallel to its Base.

First, either by Theorem 15. Page 413. Or Theorem 16 Page 414. find the proposed Frustum's Solidity in Cubick Inches. Then Divide that Content in Cubick Inches by 282. Or 231. &c. and the Quotient will be the Content of the Frustum in their respective Gallons.

But from the aforesaid Theorem 15. there may be easily deduced the following General Rule, for finding the Content of the like Frustum of any Pyramid, what Form soever its Bases are of (supposing them to be Parallel) whether they are alike or unlike.

Rule { First find the Area of each Base (viz the Top and Bottom Area's of the proposed Frustum) Then find a Geometrical Mean between those Two Area's; (per Lemma 1, Page 77.) the Sum of those Two Area's, and their Mean, being Multiplied into One Third of the Frustum's Height, will produce the Content required,

Example

Example. Suppose a Tun in the Form of the Lower Frustum of a Pyramid, whose Bases are Equilateral Triangles. Let the Side of the Top be 42 Inches; the Side of the Bottom be 63,4 Inches; And its Height (viz. Depth) be 33 Inches; What will the Content of that Tun be in Ale Gallons?

First find the Area of each Base in Inches; per Problem 7. Page 337. then find what those Area's are in Ale Gallons; per Problem 3. Page 430. Multiply those Two Area's together; the Square Root of their Product will be the Mean Area, &c. As in this Example.

Example. $\left\{ \begin{array}{l} \text{The Area of the Top is } 2,71 \\ \text{The Area of the Bottom is } 6,12 \\ \text{The Mean Area will be } 4,07 \end{array} \right\}$ Ale Gallons.

Their Sum is 12,90

Then $12,9 \times \frac{33}{3} = 141,9$. Or $\frac{12,9}{3} \times 33 = 141,9$ the Content required.

Problem 11.

To find the Content of any Right Cylinder in Gallons.

That is, To compute the Content of any Round Tun, &c. whose Diameters at Top and Bottom are Equal, and at Right Angles with its Sides.

The Content of such a Tun may be found per Theorem 11. Page 409. Or otherwise by the following Rule

Rule $\left\{ \begin{array}{l} \text{Multiply the Square of the Diameter into the Height;} \\ \text{and Divide the Product by } 359,05 \text{ (Or Multiply with} \\ 0,002785) \text{ \&c. As in Page 433. that Quotient (Or} \\ \text{Product) will be the Content required.} \end{array} \right.$

Exam. Suppose the Diameter be 42,5 and the Height 31,5 Inches. First $42,5 \times 42,5 = 1806,25$. And $1806,25 \times 31,5 = 56896,875$. Then $359,05 \mid 56896,875$ (158,46 the Content in A. Gal. &c.

Problem 12.

To find the Content of any Cone or Round Pyramid in Gallons.

Because every Cone is One Third of its Circumscribing Cylinder, See Theorem 13. Page 410. Therefore its Content may be truly found by the following Rule.

Rule $\left\{ \begin{array}{l} \text{Multiply the Square of the Diameter of its Base, into} \\ \text{the Pependicular Height; then Divide their Product} \\ \text{by } 1077,15 = 359,05 \times 3 \text{ for Ale Gallons; Or by} \\ 882,36 = 294,12 \times 3 \text{ for Wine Gallons, \&c. and the} \\ \text{Quotient will be the Content required.} \end{array} \right.$

Or

Or if the said *Product* be Multiplied with $0,000928 = \frac{0,002785}{3}$
 or with $0,001133 = \frac{0,0034}{3}$ those *Products* will be the *Content*
 in their respective *Gallons*.

Example. Suppose the *Diameter* of the *Base* be 42,5. and the
Perpendicular Height be 31,5 *Inches*; What will the *Content* be
 in *Ale Gallons*? (As before.
 First $42,5 \times 42,5 = 1806,25$. And $1806,25 \times 31,5 = 56896,875$
 Then $1077,15 \mid 56896,875$ (52,82. Or $56896,25 \times 0,000928$
 $= 52,82$ the *Content* in *Ale Gallons*. And so on for *Wine* or
Corn Gallons.

Problem 13.

To find the *Content* of the *Lower Frustum* of any *Cone*, in *Gallons*.

That is, To compute the *Content* of any *Round Tun*, &c.
 whose *Diameters* at *Top* and *Bottom* are *Parallel*, but *Unequal*.

The *Content* of such a *Tun* may be found by the *Rule* at
Problem 10. but from *Theorem 16.* Page 414. it will be *Easie* to
 deduce this following *Rule*.

Rule { To the *Triple Product* of the *Top* and *Bottom Diameters*,
 Add the *Square* of their *Difference*; Multiply that *Sum*
 into the *Height* (or *Depth*) then Divide the *Last Product*
 by 1077,15 for *Ale Gallons*, Or by 882,36 for *Wine*
Gallons; the *Quotient* will be the *Content* required.

Example. Suppose the *Diameter* at the *Top* be 52,4 *Inches*;
 the *Diameter* at the *Bottom* 44,6; and the *Height* 30 *Inches*.

if $52,4 \times 44,6 = 2337,04$. And $2337,04 \times 3 = 7011,12$ } Add
 Also $52,4 - 44,6 = 7,8$ And $7,8 \times 7,8 = 60,84$ }

The *Height* $30 \times 7071,96 = 212158,8$

Then $1077,15 \mid 212158,80$ (196,96 } the *Content* in *A. Gal.*
 Or $212158,8 \times 0,000928 = 196,96$ }

And so on for either *Wine* or *Corn Gallons* as *Occasion* requires.
 But if the *Tun* (or *Vessel*) be not truly *Circular*; That is, if
 either its *Top* or *Bottom* (or both of them) be *Elliptical*, whe-
 ther they are *alike* or *unlike* it matters not, the *Content* of such
 a *Tun* may be truly found by the *General Rule* at *Problem 10*.

Problem 14.

The *Axis* or *Diameter* of any *Sphere* or *Globe* being given (in
Inches) To find its *Content* in *Gallons*.

Every *Sphere* is Two Thirds of its *Circumscribing Cylinder*, per
Theor. 18. Page 417. from whence and *Theorem 20.* Page 420. it is
 proved,

proved, that if the *Cube* of the *Axis* of any *Sphere* (taken in *Inches*) be *Multiplied* into 0,5236. the *Product* will be the *Content* of the *Sphere* in *Cubick Inches*. Consequently, if that *Content* be *Divided* by 282. Or by 231 &c. the *Quotient* will be the *Content* in *Gallons*.

But those *Two Works* of *Multipling* with 0,5236. and then *Dividing* by 282. Or by 231 &c. may be contracted into *One*. Thus 282) 0,5236 (0,001856 will be a *Multiplia.* for *A. G.*

And 231) 0,5236 (0,002266 will be a *Multiplia.* for *W. G.*
Or 0,5236) 282 (538,57 will be a *Divisor* for *Ale Gal.*
And 0,5236) 231 (441,17 will be a *Divisor* for *Wine Gal.*

From hence arises this following *Rule*.

Rule { If The *Cube* of the *Axis* of any *Sphere*, be *Divided* by 538,57 (or *Multiplied* with 0,001856) Or *Divided* by 441,17. (Or else *Multiplied* with 0,002266) the *Quotient* (or *Product*) will be the *Sphere's Content* in their respective *Gallons*.

Example. Suppose the *Axis* or *Diameter* of a *Sphere* or *Globe* be 22 *Inches*; How many *Ale Gallons* may it hold?
Then $22 \times 22 \times 22 = 10648$. And 538,57) 10648 (19,76 *A. G.*
Or $10648 \times 0,001856 = 19,76$ *A. Gal.* the *Content* required.
And so for either *Wine*, or *Corn Gallons*, if *Occasion* require.

Problem 15.

To find the *Content* of any *Segment* of a *Sphere*, in *Gallons*.

In the *Scholium* Page 418. there are *Two Theorems* for *Resolving* of this *Problem*, according to the *Data*.

1. If the *Diameter* of the *Segment's Base*, and its *Height* are given; the *Content* may be found by the first of those *Theorems*; which gives this *Rule*.

Rule 1. { To the *Triple Square* of Half the *Diameter*, Add the *Square* of the *Height*; Then *Multiply* that *Sum* into the *Height*, and *Divide* the *Product* by 538,57 for *A. G.* Or by 441,17 for *W. G.* &c. As above.

2. But if the *Axis* of the *Sphere*, and the *Height* of the *Segment* are given; the *Content* may be found by the *Second* of those *Theorems*.

Rule 2. { From the *Triple Product* of the *Axis* into the *Height*, Subtract Twice the *Square* of the *Height*; Then *Multiply* the *Remainder* into the *Height*, and *Divide* that *Product* by 538,57 &c. As in the *Last Problem*.

Either

Either of these Rules will produce the Content of the Segment in Gallons.

Example. Suppose the Diameter of the Segment's Base be 28 Inches, and its Height be 8 Inches; What may it contain in Ale Gallons?

First 2) 28 (14. Then (per Rule 1.) $14 \times 14 \times 3 = 588$. And $6 \times 6 = 36$. Next $588 + 36 = 624$. Again $624 \times 6 = 3744$. Lastly 538,57) 3744 (6,95 the Content required.

Note, This Problem may be of Use in Gauging the Crowns of Brewer's Coppers, &c.

Section 4. The Practical Method of Gauging any fix'd Tun, Or Copper; and Making a Table to shew what it will hold at every Inch deep, usually called Inching of a Tun, &c.

First you must know, that most (if not all) Brewer's Tuns are so fix'd, as to Lean a little for Conveniency of Cleansing their Drink, which is usually called the Drip or Fall of the Tun. Now this Drip or Fall of any Tun, is the Hoof of such a Solid as that Tun is supposed to represent; and under that Consideration, it may be found as in Theorem 16. Page 414. But the Practical (and indeed the best) Way, is to Measure into the Tun (when it is Dry) so much Liquor as will just cover its Bottom; for by that Means you do not only find the true Fall, but also a true Horizontal or Level Plain over the Bottom of the Tun; from which, if the Depth of the Tun (viz. the nearest Distance from the Top of the Tun to the Surface of the Liquor) be set off upon every one of its Sides, you will then have a true Parallel Plain at the Top of the Tun, to that of the Liquor.

Then if the Sides of the Tun are Straight from the Top to the Bottom, take as many Dimensions in the aforesaid Two Plains, as are needful to find the true Area of each; And by those Two Area's, and the aforesaid Depth, find so much of the Tun's Content (per General Rule at Problem 10.) as is betwixt those Two Plains.

Next to Inch that Tun; Divide the Difference between the Top and Bottom Area's by the aforesaid Depth, and the Quotient will be an Addend or fix'd Number, which being Added to the Lesser Area, the Sum will be the Area of the next Inch, and being Added to that Area, their Sum will be the Area of the Third Inch; and so on from Inch to Inch, until the Area of every single Inch be found; the Sum of those Area's (if the Work be true) will Amount (or be Equal) to the Content found, as above. And if

the Tun's Drip or Fall be Added to the Sum of all those Area's, that Sum will be the Whole or full Content of that Tun.

Now from hence it must needs be easie to conceive, that if 1. 2. 3. or any Number of those Area's accounted from the Bottom, be Added to the Fall, that Sum will shew the Quantity of Liquor or Drink that is in the Tun, to such a Number of Wet Inches from the Bottom, as there were Area's Added together.

Or if the Sum of any Number of those Area's (being accounted from the Top) be Subtracted from the Tun's whole Content, the Remainder will shew what Quantity of Liquor or Drink is in the Tun, when there is such a Number of Dry Inches from the Top as there were Area's Subtracted.

This being well considered, it will be Easie to make a Table, either to every Wet, or Dry Inch of any Regular Tun (viz. whose Sides are Straight from Top to Bottom) what Form soever its Bases are of; and whether it stand upon the Greater or Lesser Base.

But if the Sides of the Tun are Irregular (viz. not Straight from its Top to the Bottom) Then the best and easiest Way will be to Divide or Part the Tun into several Frustums, each of 10 Inches deep, and finding the Content of every single Frustum, by taking the Diameters in the Middle of every one of those 10 Inches; (That is, the first Diameters at 5 Inches from the Top; the second Diameters at 15 Inches from the Top, &c.) and Multiplying their respective Area's with 10 (which is done by only removing the separating Comma's one Place forward to the Right Hand) If the Sum of all those Frustums be Added to the Fall (as before) that Sum will be the whole Content of the Tun.

Note, If you take the Height of the aforesaid 10 Inch Frustums in the Side of the Tun, you must allow for the Difference between the Slant Height, and the Perpendicular Height in every Frustum.

Lastly, If from the whole Content of the Tun, you Subtract the Mean Area of the first Frustum Ten times; And from the Remainder Subtract the Mean Area of the second Frustum Ten times; And from the Last Remainder Subtract the Mean Area of the Third Frustum, &c. until there Remain nothing but the Fall or Hoof of the Tun; you will then by that Means have a Table, that will shew what Quantity of Drink is in the Tun, to any Number of Dry Inches.

And this is also the Method of Gauging and Inching of Brewer's Coppers, viz. by first Measuring into the Copper so much Liquor as will just cover its Crown; then Dividing its Perpendicular Height into Frustums, and its Sides into four Equal Parts, that so Cross Diameters may be taken in the Middle of each Frustum:

But

But if the Copper be much *Wider* at the *Top* than at the *Bottom*, and its *Sides Spheroidal* or *Arching*, as generally all *Large Cop-pers* are; Then instead of taking those *Mean Diameters* in the *Middle* of every *Ten Inches*, as above; you must take them in the *Middle* of every *Six Inches*; And proceed on as before.

Now the *Quantity* of *Liquor* that would cover the *Crown* of the *Copper*, may be found without *Measuring* it as above: In order to that I do suppose the *Crown* to be the *Segment* of a *Sphere*, and the *Lower Part* of the *Copper* wherein the *Crown* *Riseth*, to be the *Frustrum* of a *Parabolick Conoid*: Then if the *Diameter* at the *Top* of the *Crown*, and its *Perpendicular Height* are given, the *Quantity* of *Liquor* may be found by this follow-
ing *Rule*.

Rule { From the *Area* of the *Plain* at the *Top* of the *Crown*,
Subtract $1\frac{1}{3}$ of the *Area* of the *Crown's Height*; the *Re-*
mainder being *Multiplied* into *Half* the *Height* of the
Crown, will produce the *Quantity* or *Number* of *Gallons*
that will cover the *Crown*.

This *Rule* is deduced from *Scholium* Page 418. and *Theorem* 15.
Page 424.

Section 5. To compute the *Content* of any *Close Cask*, in *Gallons*;
viz. of any *Butt*, *Pipe*, *Hogshead*, *Barrel*, &c.

In order to perform this difficult *Part* of *Gauging*; the *Three*
following *Dimensions* of the proposed *Cask* must be truly taken
in *Inches*, and *Decimal Parts* of an *Inch*.

Viz. { The *Bouldge* or *Bung Diameter*, within the *Cask*.
Either of the *Head Diameters*, supposing them both *Equal*.
And the *Length* of the *Cask* within.

Note, In taking these *Dimensions*, it must be *Carefully* observed;

1. That the *Bung-hole* be in the *Middle* of the *Cask*; Also, that
the *Bung-staff*, and the *Staff* over against the *Bung-hole*, are both
Regular or *Even* within.

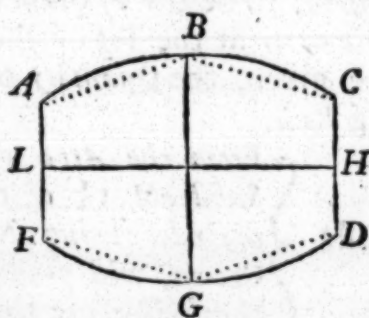
2. That the *Heads* of the *Cask* are *Equal* and truly *Circular*; if
so, the *Distance* between the *Inside* of the *Chine*, to the *Outside* of
its opposite *Staff*, will be the *Head Diameter* within the *Cask*, very
near.

3. With a *sliding Pair* of *Calipers* (made on purpose for that
Use) take the *shortest Distance* or *Length* between the *Out-sides* of
the *Two Heads* (supposing them even;) from that *Length* Sub-
tract $1\frac{1}{2}$ *Inch* (More or Less, according to the *Largeness* of

the Cask) for the Thickness of the Head; The Remainder will be the Length of the Cask within.

Now by these Dimensions, one would suppose the Content of the Cask were perfectly Limited; But it will be Easie to perceive by the following Figure, that the Diameters (abovesaid) and the Length of one Cask, may be Equal to those of another; and yet one of those Casks may contain or hold several Gallons more than the other.

As for Instance, Suppose the Annexed Figure *ABCDGF*, to represent a Cask; Then it is plain, that if the Outward Curved Lines *ABC*, and *FGD* are the Bounds or Staves of the Cask; it must needs hold more than if the Inner Straight or prick'd Lines were its Bounds or Staves; And yet the Bung Diameter *BG*, Head Diameters *CD* and *AF*, And the Length *LH* are the same in both those Casks.



Whence it plainly appears, that no one Certain or General Rule can be prescribed, to find the true Content of all sorts of Casks; And therefore Gaugers do usually suppose every Cask to be in the Form of some one of these following Solids.

- Viz. {
- I. The Middle Zone or Frustum of a Spheroid.
 - II. The Middle Zone or Frustum of a Parabolick Spindle.
 - III. The Lower Frustums of Two Equal Parabolick Conoids.
 - IV. The Lower Frustums of Two Equal Cones.

Now the Way of Guessing at the Cask's Form, and computing its Content according to that supposed Form; I shall here shew in their Order.

I. If the Staves of the Cask are very Curved or Arching (as the outward Lines of the Last Figure) Then the Cask is supposed to be in the Form of the Middle Zone or Frustum of a Spheroid; whose Content may be computed, per Theorem 22. Page 421. which gives these Two Rules.

Rule I. { To Twice the Square of the Bung Diameter, Add the Square of the Head Diameter; Multiply that Sum into the Length, and Divide the Product by 1077,15.
 Viz. $3,8197 \times 282$ for Ale Gallons; and by 882,36.
 Viz. $3,8197 \times 231$ for Wine Gallons. Or thus,

Rule

Rule 2. { To Twice the Area of the Bung Circle, Add the Area of the Head Circle; Multiply their Sum into one Third of the Length, and the Product will be the Content in their respective Gallons.

Example 1. Suppose a Cask in the Form of the Middle Zone of a Spheroid, whose Bung Diameter is 31,5 Head Diameter 24,5 And its Length 42 Inches.

First $31,5 \times 31,5 \times 2 = 1984,5$ And $24,5 \times 24,5 = 600,25$

Again $1984,5 + 600,25 = 2584,75$. And $2584,75 \times 42 = 108559,5$

Then $1077,15$) $108559,5$ ($100,78$ the Content in Ale Gallons.

And $882,35$) $108559,5$ ($123,03$ the Content in W. Gallons.

Or thus by the Second Rule.

Bung Diameter 31,5 Twice its Circle's Area is 5,5270

Head Diameter 24,5 its Circle's Area is 1,6718

The Length 42 Divided by 3. is 14. $7,1988 =$ their Sum.

Then $7,1988 \times 14 = 100,78$ the Content in A. Gallons. As before.

And so the Content in Wine Gallons may be found.

II. If the Staves of the Cask are not quite so much Curved or Arching as was supposed before; the Cask is then taken for the Middle Frustum of a Parabolick Spindle; And its Content is computed, as per Theorem 27. Page 426. which gives this Rule.

Rule { To Twice the Square of the Bung Diameter; Add the Square of the Head Diameter; from their Difference, Subtract Four Tenths of the Square of the Difference of the Diameters; Multiply the Remainder into the Length, and Divide the Product by 1077,15 &c. As above.

Example 2. Suppose the Dimensions the same as before. Then

$31,5 \times 31,5 \times 2 : + 24,5 \times 24,5 = 2584,75$. And $31,5 - 24,5 = 7$

Again $7 \times 7 \times 0,4 = 19,6$. And $2584,75 - 19,6 : \times 42 = 107736,3$

Then $1077,15$) $107736,3$ ($100,01$ the Cont. in A. G. &c. for W. G.

III. When the Staves of the Cask are but very little Curved or Arching, Then it's supposed to be in the Form of the Frustums of Two Equal Parabolick Conoids, abutting or joyning together upon one common Base at the Bouldge; and the Content may be found per Theorem 25. Page 424. which gives these Rules.

Rule

Rule 1. { To the Square of the Bung Diameter, Add the Square of the Head Diameter; Multiply their Sum into the Length, and Divide the Product by 718,08
(viz. $2,5464 \times 282$) for Ale Gallons; Or by 588,22 (viz. $2,5464 \times 231$) for Wine Gallons. Or thus,

Rule 2. { To the Area of the Bung Circle, Add the Area of the Head Circle; Multiply the Sum into Half the Length, and the Product will be the Content required.

Example 3. with the same Dimensions as before. Then $31,5 \times 31,5 : + 24,5 \times 24,5 = 1592,5$ And $1592,5 \times 42 = 66885$ And 718,08) 66885 (93,01 the Content in Ale Gallons.

Or 588,22) 66885 (113,7 the Content in Wine Gallons:

IV. If the Staves of the Cask are Straight from the Bouldge to the Head, as the Inner prick'd Lines in the Last Figure; (if such a Cask can be made) it is then taken for the Lower Frustums of Two Equal Cones, abutting or joyning together upon one common Base at the Bouldge. And its Content may be computed as at Problem 13. Page 439. Or per Theorem 15. Page 413. Thus,

Rule { To the Sum of the Squares of the Head and Bung Diameters, Add their Product; then Multiply that Sum into the Length, and Divide the Last Product by 1077,15. Or by 882,36. The Quotient will be the Content, &c.

Example 4. With the same Dimensions as before.

First $31,5 \times 31,5 : + 24,5 \times 24,5 : + 31,5 \times 24,5 = 2364,25$ And $2364,25 \times 42 = 99298,5$ Then 1077,15) 99298,5 (92,18 the Content in Ale Gallons. And so on for Wine Gallons.

Thus you have the Methods of computing the true Contents of the four Solids, in whose Forms all Casks are supposed to be. And by the Examples it appears, that Four such Casks as have their Dimensions all Equal, and the same with those above mentioned, their Contents will be as in the Margin.

	Ale Gallons.	Differ.
I.	100,78	
II.	100,01	0,77
III.	93,01	7,00
IV.	92,18	0,83

From the Disproportion or Inequality of these Differences, it will be Easie to Conceive, That there may be several Casks whose Contents cannot be truly found, according to the aforesaid supposed Forms: And therefore, in order to Rectify the said Inequalities, some Authors (that have written upon this Subject) have laid down Theorems of their own Invention (and yet called them by these

these Names) Others have proposed Tables for the same Purpose : But since it is so, that we can only Guess at the Truth, the Plainest and Easiest way is to be preferr'd in Practice. And that is, by finding such a Mean Diameter as will Reduce the proposed Cask to a Cylinder.

Thus { Multiply the Difference between the Head and Bung Diameters with 0,7. Or with 0,65 Or with 0,6 Or with 0,55 according as the Staves of the Cask are More or Less Arching ; Add the Product to the Head Diameter, and the Sum will be the Mean Diameter required. Then find the Content as at Prob. II. Page 438.

Example. With the same Dimensions as before. Then the Bung Diameter Less the Head Diam. is $31,5 - 24,5 = 7$. And

	M. D.	A. G.	Cont.	Dif.
24,5 + {	$7 \times 0,7 = 29,4$	its Area $2,4073 \times 42 = 101,10$		2,39
	$7 \times 0,65 = 29,05$	— $2,3504 \times 42 = 98,71$		2,36
	$7 \times 0,6 = 28,7$	— $2,2941 \times 42 = 96,35$		2,32
	$7 \times 0,55 = 28,35$	— $2,2385 \times 42 = 94,03$		

From these it may be observed, that the Difference between each Cask's Content is Regular, and very near Equal ; which plainly shews, that there is not so much Room left of Error this way of Computing their Contents, as was by the aforesaid Forms.

Now the First of these Four (viz. with 0,7) is very commonly used amongst Gaugers for all Sorts of Casks ; But I did never Gauge any Cask that would Contain quite so much as that Rule did make it : And the Reason doth appear very plain from Theorem 22. Page 421. being compared with Theorem 19. Page 420. and the Last Figure. viz. that no Cask (being Regularly made) can hold More than the Middle Frustum of a Spheroid. But I always found by Experience, that if the Second and Third of these Rules (viz. with 0,65 and 0,6) were duly Applied, they would answer very near the Truth amongst the common sort of Casks ; And the Fourth Rule (viz. with 0,55) will come pretty near the Truth in computing the Contents of Casks whose Staves are almost Straight betwixt the Head and the Bung. viz. such as Wine Pipes, &c.

Section 6. To find what Quantity of Liquor, is either Drawn forth, Or Remaining in, any Spheroidal Cask ; Usually called the Ullage of a Cask. Which bath Two Cases.

Case 1. To find what Quantity of Liquor is in the Cask, when its Axis is Perpendicular to the Horizon ; viz. when it stands upright upon one of its Heads, In

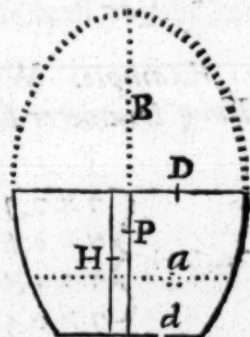
In order to perform this the *Easiest Way*, it will be Convenient to know how to Calculate the Area of any Circle betwixt the Bung and Head, whose Distance from the Bung or Middle of the Cask is given. Now that may be done by this Proportion.

Viz. $\left\{ \begin{array}{l} \text{As the Square of Half the Length of the Cask : Is to the} \\ \text{Difference between the Bung and Head Area's :: So is the} \\ \text{Square of any Circle's Distance from the Bung : To the Dif-} \\ \text{ference between the Bung Area and the Area of that Circle ;} \\ \text{viz. the Area of the Liquor's Surface.} \end{array} \right.$

Demonstration.

Let $\left\{ \begin{array}{l} H = \text{Half the Length of the Cask,} \\ D = \text{Half the Bung Diameter.} \\ d = \text{Half the Head Diameter.} \end{array} \right.$

And $\left\{ \begin{array}{l} P = \text{the Distance of any Circle from} \\ \text{the Bung.} \\ a = \text{Half the Diameter of that Circle.} \end{array} \right.$



Then, according to the common Property of the Ellipsis, Page 362. it will be,

$$BB : DD :: BB - HH : dd. \text{ And } BB : DD :: BB - PP : aa.$$

$$\text{Ergo } \left\{ \begin{array}{l} \frac{DDHH}{DD - dd} = BB. \text{ And } \frac{DDPP}{DD - aa} = BB. \end{array} \right.$$

$$\text{Consequently, } \left\{ \frac{DDHH}{DD - dd} = \frac{DD - aa}{DDPP} \right.$$

This Equation being brought out of the Fractions will become $DDHH - aaHH = DDPP - ddPP$.

Which gives this Analogy $HH : DD - dd :: PP : DD - aa$. Then $DD - aa$ being Subtracted from DD , will leave aa ; But Circle's Area's are in Proportion to the Squares of their Diameters; per Theorem 6. Page 401. Therefore, &c. Q. E. D.

Then, from the Bung Area, Subtract One Third Part of the aforesaid Difference; viz. between the Bung Area, and the Area of the Liquor's Surface; Multiply the Remainder with the Liquor's Distance from the Bung, and the Product will shew what Quantity of Liquor is either above, or under Half the Content of the Cask.

Example. Let us suppose a Cask of the same Dimensions with that in the First Example, Page 445. And let it be required to find what Quantity of Liquor is in it, (of Ale Measure) when there is but 9 Inches Wet. Here Half the Length of the Cask

Inches, whose Squares is 441 and the Liquor's Distance from the Bung is $21 - 9 = 12$. its Square is 144. the Difference between the Bung and Head Area's is 1,0917 ($= 2,7635 - 1,6718$.) Then $441 : 1,0917 :: 144 : 0,3564$.

And $2,7635 - 0,3564 = 2,4071$ the Area of the Liquor's Surface.

Again 3) $0,3564$ ($0,1188$. And $2,7635 - 0,1188 = 2,6447$ Then $2,6447 \times 12 = 31,7364$ what the Cask wants of being Half full. Consequently $50,39 - 31,73 = 18,66$ will be the Quantity of Liquor in the Cask at 9 Inches Wet, in Ale Gallons.

And if the Cask had wanted but 9 Inches of being Full; Then $50,39 + 31,73 = 82,12$ would have been the Quantity of Liquor in the Cask.

Note, Because the Two First Terms (viz. 441. and 1,0917) in the Proportion, are fix'd, viz. continue the same for any Distance; it will be very Easie to Calculate the Area's of all the Circles betwixt the Bung and Head to every Inch; And by that Means to make a Table that will shew what Quantity of Liquor, is either Drawn Out, or Remaining In the Cask at any Depth.

Case 2. To find what Quantity of Liquor is in any Cask, when its Axis is Parallel to the Horizon: viz. when it Lies along.

There are Variety of Tables to be found in Books of Gauging for this Purpose: But I always observed, that the following Method of computing the Ullage by a Table of the Segments of a Circle, came very near the Truth in all Sorts of Casks, which is thus performed.

1. By the Bung and Head Diameters, find such a Mean Diameter as you judge will Reduce the proposed Cask to a Cylinder; by the Method Laid down in Page 447. and then find its full Content, as in those Examples.

2. From the Bung Diameter, Subtract the Mean Diameter; and Halve their Difference, (viz. Divide it by 2.)

3. From the Wet Inches of the proposed Ullage, Subtract the said Half Difference; and call it x ; then observe this Proportion:

{ As the Mean Diameter : Is to 100 (the Diameter of
Viz. { the Tabular Circle) :: So is the Last Difference (viz. x)
To a Versed Sine in the Table. (Page 435.)

Then if the Tabular Segment, which stands against that Versed Sine, be Multiplied into the Content of the Cask: the Product will shew the Ullage; viz. what Quantity of Liquor is either in the Cask, or Drawn forth.

M m m

Example

Example 1. Let the Cask be that of the Second Sort in Page 447. viz. whose Bung Diameter is 31,5 Inches ; Mean Diameter 29,05 ; and Content 98,71 Ale Gallons ; And suppose there were 10,5 Inches Wet in it ; It is required to find the Wet and Dry Gallons ?

Here $31,5 - 29,05 = 2,45$ its Half is 1,22 And $10,5 - 1,22 = 9,28$

Then $29,05 : 100 :: 9,28 : 0,319 = V. Sine$; Its Segm. is 0,2748

And $98,71 \times 0,2748 = 27,12$ the Number of Wet Gallons.

Again $31,5 - 10,5 = 21$ the Dry Inches ; And $21 - 1,22 = 19,78$

Then $29,05 : 100 :: 19,78 : 0,68$; Its Segment is 0,7241

And $98,71 \times 0,7241 = 71,48$ the Number of Dry Gallons.

Proof $71,48 + 27,12 = 98,6$ the Content of the Cask very near ; which plainly shews the Truth of this Method:

Thus far may suffice concerning Gauging of Backs or Coolers, Tuns, Coppers and Casks, &c. To which I shall only Add ; That, as the Contents of all Brewer's Utensils are to be Computed by the Ale Gallon : So the Contents of all Distiller's Utensils, (viz. all their Wash Backs, Stills, and Casks, &c.) must be Computed by the Wine Gallon.

And in Gauging of Malt, (upon which there is now a Duty of four Shillings per Bushel) you must Observe, That a Corn or Malt Bushel doth contain 2150,42 Cubick Inches (See Page 36.) And therefore, in Gauging of Malt-Cisterns, or other Vessels, 2150,42 will be a constant or fix'd Divisor for finding the Area's of all Right-lin'd Figures in Bushels, at one Inch Deep ; and 2738 will be a constant or fix'd Divisor, for finding the Area's of Circular Figures.

I have Omitted the Business of Gauging Mash-Tuns, and taking an Account of the Goods or Grains, in order to Estimate what Quantity of Worts were produced from them, &c. because I could never find (by all my Observations) any Certainty therein ; nor is it possible there should be any, by Reason of the Great Difference that is in Malt, (and its Grinding too ;) for the Best Malt (well Ground) will yield or produce the Most Worts, and Least Grains ; on the contrary, Bad Malt (being ill Ground) yields the Least Worts, and Most Grains.

Note, Whereas I seem wholly to Explode the Use of Sliding Rules, in Page 427. I mean only Pocket Rules, viz. such as are of Nine Inches or a Foot Long, whose Radius of the Double Line of Numbers is not Six Inches ; And therefore the Graduations or Divisions of those Lines are so very Close, that they cannot well be

be Distinguished. But when the Rules are made Two, or Three Foot Long (I had One of Six Foot) then they may be of some Use, especially in Small Numbers; altho' even Then, the Operations may be much better done by the Pen; for indeed the Chief Use of Sliding Rules, is in taking of Dimensions; and for that Purpose they are very Convenient.

But here I must not omit to Recommend a Five Foot Rule, Composed of Six Parts or Legs (viz. Ten Inches each Part) with Brass Joints, put together with Steel Screws (which I Contrived and Made many Years ago) the Last Leg (viz. one of the Extream Legs) having a Sliding Part put to it, and the Whole, Decimally Divided into Inches, &c. to which let there be adjoyned only Two Lines of Circle's Area's; viz. one for Ale Gallons, and another for Wine Gallons, placed on each Side of the Line of Inches. Such a Rule is not only very Useful for taking of any Dimensions between Ten Inches and Sixty; but, you have by the afore-said Lines, the Area of any Circle (whose Diameter is given in Inches, &c.) more Ready than by the Table of Area's (which are usually cramb'd in every Book of Gauging) And not only so, but you may also take the Area of any Circle, instead of taking its Diameter in Inches; Which I always found very Useful and Ready in Practice, both in Gauging of Worts in Open Vessels or Tubs, &c. and in Gauging of Casks; all which the Practicall Gauger will soon perceive; to whom I wish Good Success.

F I N I S.

ADVERTISEMENT.

THE abovesaid Rules, and all Sorts of Mathematical Instruments, are very Accurately made by Mr. John Rowley, (mention'd before in Page 369) at the Sign of the Globe under St. Dunstan's Church in Fleet-Street, London; Where Surveyors of Land, &c. may be furnished with a New Sort of a Protractor, by which any Right-lin'd Angle may be Projected or Actually laid down to a Minute; the Making whereof I Invented, and Communicated to the said Mr. Rowley only.

If the Reader were but *Sensible* of the *Great Care* and *Difficulty* that unavoidably attends *Correcting* the *Press* to *Books* of this *Nature*; he would the more readily *Excuse* and *Amend* the following *Errata's*.

Those that are marked with (*) are *Errors* in the *Margin* of the *Algebra Work*.

E R R A T A.

Page.	Line.	Read.	Page.	Line	Read.
8	28	2651	154	28	48a-96
27	7	71092	163	17	8bbb
36	36	17203,2	172	22	$\sqrt[3]{ac-ba}$
45	2	21 <i>gra.</i>	193	38	blot out : d
49	11	$\frac{3}{4}$ of $\frac{2}{3}$	200	25	(a + 27,5 = 0
54	9	$\frac{1}{4}$	215	26	Page 201
55	24	As in these	216	11	Or 34 ⁹
56	21	$\frac{102}{3}$	224	4	9 - 78a (*)
<i>Ditto</i>	—	$\frac{1}{3}) \frac{102}{3} (= 6$	<i>Ibid</i>	7	12 - 72,5 (*)
61	33	are in either	227	36	-16y = 1232
63	4	0,578	239	33	,0223927
70	28	0,00104167	241	24	24691200 —
<i>Ibid</i>	37	0,00208333	272	29	1,0295630141
74	30	the Number of	274	22	Persons
75	15	Sixty three	307	18	AB to C
89	26	6212 $\frac{3}{4}$ d	323	23	17 x ss (*)
<i>Ibid</i>	27	blot out 17 s.	330	29	ssaa - 2sa ³ + a ⁴
91	6	10 $\frac{3}{4}$ d	338	32	$\square DA - \square DB =$
96	23	TgP=Gpe	367	33	= d. 3.
107	38	1l. 4s. 7d. =	380	17	zz = yy
111	11	How may	<i>Ibid</i>	18	z = y
<i>Ibid</i>	30	at . 4 . 3	402	27	Suppose D = 32
113	2	15 : 12	432	ult	) 268,800000 (

 Note, at *Question 5*. *Page 93*. the *Operation* or *Work* is all *Mistaken*; for the *Answer* should be only *13 Ounces*.

